This is a preliminary list of problems for the mini-course *Synchronization in Random Graphs*, taught at PISA 2026 in Santiago de Chile. It is likely to be plenty of errors, typos, etc. Please do not hesitate to contact me at pgroisma@dm.uba.ar if you find any of them. ´

1. **Phase cohesiveness and arc length.** Pick $\gamma < 2\pi/3$ and $n \geq 3$. Show the following statement: if $\theta \in \mathbb{T}^n$ satisfies $|\theta_i - \theta_j| \leq \gamma$ for all $i, j \in \{1, \ldots, n\}$, then there exists an arc of length $\gamma$ containing all angles, that is, $\theta \in \Gamma_{\text{arc}}(\gamma)$. Show that the statement is false if $\gamma \geq 2\pi/3$.

2. **Order parameter and arc length.** Given $n \geq 2$ and $\theta \in \mathbb{T}^n$, the shortest arc length $\gamma(\theta)$ is the length of the shortest arc containing all angles, i.e., the smallest $\gamma(\theta)$ such that $\theta \in \Gamma_{\text{arc}}(\gamma(\theta))$. Given $\theta \in \mathbb{T}^n$, the order parameter is the centroid of $(\theta_1, \ldots, \theta_n)$ understood as points on the unit circle in the complex plane $\mathbb{C}$:

$$r(\theta)e^{\psi(\theta)} := \frac{1}{n}\sum_{j=1}^{n} e^{\mathbf{i}\theta_j}$$

   where recall $\mathbf{i} = \sqrt{-1}$. The order parameter magnitude $r$ is known to measure synchronization. Show the following statements: Show that

   (i) if $\gamma(\theta) \in [0, \pi]$, then $r(\theta) \in [\cos(\gamma(\theta)/2), 1]$.

   (iii) all oscillators are phase-synchronized if and only if $r = 1$, and

   (iv) if all oscillators are spaced equally on the unit circle, then $r = 0$.

3. **The Cycle $C_n$.** Consider a Kuramoto oscillator network defined over a symmetric cycle graph with identical unit weights and zero natural frequencies. The equilibria are determined by

$$0 = \sin(\theta_i - \theta_{i-1}) + \sin(\theta_i - \theta_{i+1}),$$

   where $i \in \{1, \ldots, n\}$ and all indices are evaluated modulo $n$.

   (a) Show that for $n > 4$ and $-\frac{n}{2} < q < \frac{n}{2}$, $\theta^q = (0, \frac{2q\pi}{n}, \frac{4q\pi}{n}, \ldots, \frac{(n-1)2q\pi}{n})$ is an equilibrium.

   (b) Show that if $-\frac{n}{4} < q < \frac{n}{4}$, then $\theta^q$ is a local minimum of the energy.

4. **Potential and order parameter.** Recall $U(\theta) = \sum_{\{i,j\}\in E} a_{ij}(1 - \cos(\theta_i - \theta_j))$. Prove $U(\theta) = \frac{Kn}{2}(1 - r^2)$ for a complete homogeneous graph with coupling strength $a_{ij} = K/n$.

   **Note:** Recall the Courant-Fisher minimax characterization of the eigenvalues of a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$:

$$\lambda_k = \min_{S\in S_k} \max_{\substack{u\in S \\ \|u\|_2=1}} \langle Pu, u\rangle,$$

   where $S_k$ is the set of $k$-dimensional vector subspaces of $\mathbb{R}^n$.

5. Let $L$ be the Laplacian matrix of the graph $G = (V, E)$ with (symmetric) adjacency matrix $A$ and real eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Prove

   (a) For all $u \in \mathbb{R}^n$,
   $$\langle Lu, u\rangle = \sum_{\{i,j\}\in E} a_{ij}(u_j - u_i)^2.$$

   (b) $L\mathbf{1}_n = 0$.

   (c)
   $$\lambda_2 = \min_{|u|_2=1, u\perp \mathbf{1}_n} \langle Lu, u\rangle$$

   (d) $\lambda_1 = 0$ and ($\lambda_2 > 0$ if and only if $G$ is connected).

   (e) $L$ is positive semidefinite. If $G$ is connected, $L\,|_{\langle \mathbf{1}_n\rangle^\perp}$ is positive definite. That is $\langle Lu, u\rangle > 0$ for every nonzero $u \in \langle \mathbf{1}_n\rangle^\perp$

   (f) If $A' \geq A$ component wise, then $\lambda_k(A') \geq \lambda_k(A)$.

6. Prove that the algebraic connectivity of the given graphs is the given one.

| **Graph** | **Algebraic connectivity** |
|---|---|
| path graph $P_n$ | $2(1 - \cos(\pi/n)) \sim \pi^2/n^2$ |
| cycle graph $C_n$ | $2(1 - \cos(2\pi/n)) \sim 4\pi^2/n^2$ |
| star graph $S_n$ | 1 |
| complete graph $K_n$ | $n$ |

7. **Upper and lower bound on largest Laplacian eigenvalue.** Let $G$ be an undirected graph with symmetric Laplacian matrix $L = L^T \in \mathbb{R}^{n\times n}$, Laplacian eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and maximum degree $d_{\max} = \max_{i\in\{1,\ldots,n\}} d_i$. Show that the maximum eigenvalue $\lambda_n$ satisfies:
   $$d_{\max} \leq \lambda_n \leq 2d_{\max}.$$
   Hint: For the upper bound you may want to use Gershgorin Disks Theorem, or write $L = 2D - (D + A)$.

8. Let $y(t)$ be a smooth function that verifies for $0 \leq t \leq T$,
   $$\dot{y}(t) \leq -cy(t),$$
   for some $c > 0$. Then $y(t) \leq y(0)e^{-ct}$ for all $0 \leq t \leq T$.

9. Prove that for smooth $u$,
   $$\lim_{\epsilon\to 0} \frac{1}{\epsilon^d} \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \left(\frac{1 - \cos(u(y) - u(x))}{\epsilon^2}\right) \mathbf{1}\{|y - x| < \epsilon\}\, dy\, dx = \kappa_d \int_{\mathbb{T}^d} |\nabla u(x)|^2\, dx.$$

10. Let $N \sim \mathrm{Bi}(n, p)$. Assume $p \to 0$ as $n \to \infty$.

    (a) Prove that
    $$\mathbb{E}(e^{-tN}) \leq e^{-tnp}, \qquad \text{for } n \text{ large enough.}$$

(b) Prove that

$$\mathbb{P}(|N - np| > t) \le 2\exp\left\{-\frac{t^2}{2np + \frac{2}{3}t}\right\}$$

Hint.: Use Bernstein inequality.

(c) Take $t = \delta np$ in the above inequality.

11. **Poissonization.** Let $x_0, x_1, \dots$ be i.i.d. uniform random variables on the torus $\mathbb{T}^d$. Let $G_n$ by a RGG with parameter $\epsilon_n$ constructed with $x_0, \dots, x_n$ and $U_n$ the Kuramoto energy of $G_n$.

$$U_n(\mathbf{u}) = U_n(u_0, \dots, u_{n-1}) = \frac{1}{2}\sum_{i=1}^{n} \sum_{\substack{j\sim i \\ x_j \in V_n}} (1 - \cos(u_j - u_i)).$$

Here $\sigma_d$ is the volume of the $d$–dimensional unit ball. Finally let $\mathsf{N}$ be a Poisson random variable of parameter $n$.

(a) Prove that for every $\delta > 0$

$$\mathbb{P}\left(\left|\frac{\mathsf{N} - n}{n}\right| > \delta\right) \le e^{-cn},$$

for some constant $c > 0$.

(b) Prove that for every $p \ge 1$

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{|\mathsf{N} - n|}{n}\right)^p = 0.$$

(c) Prove that if $u \in C^1(\mathbb{T}^d)$

$$\lim_{n\to\infty} n^{-2}\epsilon_n^{-(d+2)}\mathbb{E}|U_{\mathsf{N}}(u) - U_n(u)| = 0.$$

12. Let $N_i$ be the degree of node $i$ in a RGG on the $d$–dimensional torus $\mathbb{T}^d$ with parameter $\epsilon_n$. Prove that

$$\mathbb{P}\left(\sup_{i=1}^{n} N_i \ge \frac{2\sigma_d n\epsilon_n^d}{(2\pi)^d}\right) \le 2ne^{-c\epsilon_n^d n}.$$

13. Call $N_{ij} = |\{k : |x_i - x_k| < \epsilon_n, |x_j - x_k| < \epsilon_n\}|$ the number of common neighbors of $i$ and $j$,

(a) Prove that $\mathbb{E}(N_{ij}|i \sim j) \ge c_d n\epsilon_n^d$.

(b) Prove that

$$\mathbb{P}\left(N_{ij} \le \frac{c_d n\epsilon_n^d}{2}\middle| i \sim j\right) \le e^{-cn\epsilon_n^d}.$$

(c) Prove that

$$\mathbb{P}\left(\inf_{i\sim j} N_{ij} < \frac{c_d n\epsilon_n^d}{2}\right) \le n^2 e^{-c\epsilon_n^d n}\frac{\epsilon_n^d\sigma_d}{(2\pi)^d}.$$

14. For a RGG with parameter $\epsilon_n$, prove that if $\frac{n\epsilon_n^d}{\log n} \to \infty$ and $\epsilon_n \to 0$, then

$$\lim_{n\to\infty} \frac{\lambda_n(L)}{\lambda_2(L)} = \lim_{n\to\infty} \frac{\lambda_n(\mathcal{L})}{\lambda_2(\mathcal{L})} = \infty.$$