# Mini-course on Phase transitions in percolation and interacting particle sytems at the Summer School: Probability in South America

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January 14, 2026

The intent of this mini-course is to introduce the phenomenon of a phase transition in mathematical models as well as a handful of pivotal tools alongside standard models they can be applied to. For further results and detailed proofs of central results mentioned (but not proved) here, references will be given in the corresponding sections.

# 1 Percolation

## 1.1 Introduction

Percolation is one of the simplest models in probability theory which exhibits what is known as *critical phenomena* or *phase transition*. This usually means that there is a natural parameter in the model at which the global behavior of the system drastically changes. Percolation theory is an especially attractive subject being an area in which the central problems are easily stated but whose solutions (when they exist) often required ingenious methods. A standard reference for the field is the book by Grimmet [25]. For the study of percolation on general graphs, see [44].

In the standard model of percolation theory, so-called *bond percolation* one considers the $d$-dimensional integer lattice, which is the graph consisting of the set $\mathbb{Z}^d$ as vertex set together with an edge between any two points having Euclidean distance 1. Then one fixes a parameter $p$ and independently declares each edge of this graph to be open with probability $p$ and investigates the structural properties of the obtained random subgraph consisting of the vertices $\mathbb{Z}^d$ together with the set of open edges. The type of questions that one is interested in are of the following sort. Are there infinite components? Does this depend on $p$? Is there a critical value for $p$ at which infinite components appear? Can one compute this critical value? How many infinite components are there? A closely related model is that of *site percolation*, in which *nodes* are independently declared open (resp. closed) with a fixed probability $p$.

The study of percolation started in 1957 motivated by some physical considerations and very much progress has occurred through the years in our understanding. In the last decades in particular, there has been tremendous progress in our understanding of the two-dimensional case (more accurately, for the hexagonal lattice) due to Smirnov's proof of conformal invariance and Schramm's SLE processes which describe critical systems.

## 1.2 Bond percolation and coupling

The concept of bond percolation introduced above works the same on every (simple) graph $G = (V, E)$ (having bounded degree): For every edge $e \in E$ an independent coin is tossed and if it comes up heads (which has probability $p$), we keep the edge, otherwise it is removed from the graph. The quantity of interest (aiming at a first example of a non-trivial phase transition) is the probability that a given vertex (call it the origin $\mathbf{0}$) is contained in an infinite component in the random subgraph given by i.i.d. bond percolation on $\mathbb{Z}^d$. Commonly, this probability is denoted by $\Theta_{\mathbb{Z}^d}(p) := \mathbb{P}_p(\mathbf{0} \rightsquigarrow \infty)$. We will use the basic concept of coupling to verify that $\Theta_{\mathbb{Z}^d}(p)$ is increasing both in $p$ and $d$.

**Definition 1**
For two given distributions $\mu$ and $\nu$, a *coupling* is a pair of random variables $(X, Y)$ defined on the same probability space so that $X \sim \mu$ and $Y \sim \nu$.

For different values of $p$, we can couple the different bond percolation processes in a simple monotone way: Given a sequence $(U_e)_{e \in E}$ of independent unif$([0, 1])$ random variables, we declare the edge $e$ to be open in the bond percolation with parameter $p$ if and only if $U_e \leq p$. In this way, the corresponding random subgraphs are coupled in such a way that as $p$ increases from 0 to 1 the edge sets grow pointwise.

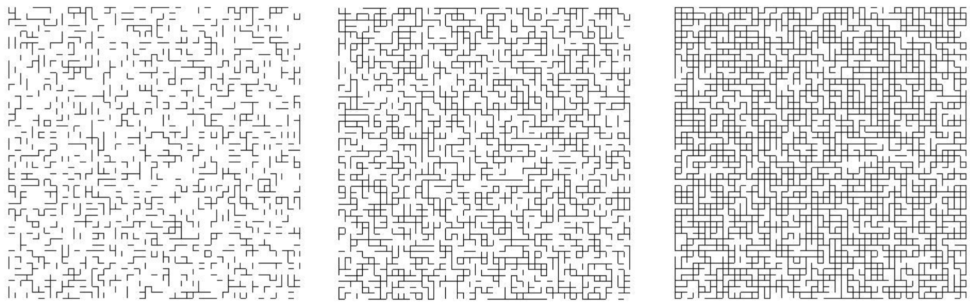

Figure 1: Configurations of bond percolation on $\mathbb{Z}^2$ for three values of the parameter $p$: 0.3 (left), 0.5 (middle) and 0.7 (right).

Conseqently for any infinite graph $G$, the probability that $p$ bond percolation contains an infinite component, $\Theta_G(p)$, is monotone in $p$. In the same way, it is monotone in the underlying graph: Since $\mathbb{Z}^d$ can be seen as a subgraph of $\mathbb{Z}^{d+1}$, we can consider the bond percolation for fixed $p$ on $\mathbb{Z}^d$ as a part of the corresponding process on $\mathbb{Z}^{d+1}$.

Next, we want to pin down how the probability $\mathbb{P}_p(\mathbf{0} \rightsquigarrow \infty)$ relates to the existence of an infinite component:

**Proposition 1.1**
*Bond percolation with parameter p on G contains an infinite component with positive probability if and only if* $\Theta_G(p) > 0$.

Apparently, only the "only if" part deserves a second thought. This is, however also rather straight forward using either union bound or alternatively a technique called *local modification*.

In fact, on a (vertex-)transitive graph $G$, Birkhoff's ergodic theorem (see below) immediately implies that the existence of an infinite component is a trivial event, i.e. has probability either 0 or 1. For those of you unfamiliar with ergodic theory, let us just mention as a side note that a system is called *ergodic*, if there exists a measure-preserving transformation $T$ such that all events $E$ whose symmetric difference with their pre-image under $T$ is a nullset are trivial, i.e. for $E$ such that $\mathbb{P}(E\Delta T^{-1}(E)) = 0$, it holds $\mathbb{P}(E) \in \{0, 1\}$. Alternatively, you can use Kolmogorov's 0-1 law here, see exercise 7.

Take a moment to verify that in one dimension, i.e. on $\mathbb{Z}$ the probability $\mathbb{P}(\mathbf{0} \rightsquigarrow \infty)$, that the origin percolates, is 0 for $p < 1$ and equals 1 in the trivial case $p = 1$. Next, we want to understand that the picture in two (and higher) dimensions is more interesting, i.e. the phase transition for bond percolation on $\mathbb{Z}^d$, $d \geq 2$ from finite components to an infinite one happening at some critical $p_c(d)$ which lies in $(0,1)$.

Let $C(v)$ denote the component containing $v \in V$ in our random graph. This is just the set of vertices connected to $v$ via a path of open edges. Of course $C(v)$ depends on the realization, but we do not write this explicitly. Note that $\mathbb{P}_p(|C(\mathbf{0})| = \infty) = \Theta(p)$.

## 1.3 The existence of a nontrivial critical value

The main result in this section is that on $\mathbb{Z}^2$, for $p$ small (but positive) $\Theta(p) = 0$ and for $p$ large (but less than 1) $\Theta(p) > 0$. In view of this (and the monotonicity of $\Theta$), there is a critical value $p_c(2) \in (0,1)$ at which the function $\Theta(p)$ changes from being 0 to being positive. This illustrates a so-called phase transition describing a change in the global behavior of a system as we move past some critical value.

The method of proof we will employ is commonly called the *first moment method*, which just means you bound the probability that some nonnegative integer-valued random variable is positive by its expected value (which is usually much easier to calculate). In the proof below, we will implicitly apply this first moment method to the number of self-avoiding paths of length $n$ starting at 0 and for which all the edges of the path are open.

**Proposition 1.2**
*For bond percolation on* $\mathbb{Z}^2$ *it holds* $\Theta(p) = 0$, *for* $p < \frac{1}{3}$.

PROOF: Let $A_n$ be the event that there is a self-avoiding path of length $n$ starting at $\mathbf{0}$ using only open edges. For any given self-avoiding path of length $n$ in the original graph, the probability that all the edges of this given path are open is $p^n$ (by independence). The number of such paths in $\mathbb{Z}^2$ is at most $4 \cdot 3^{n-1}$, since there are 4 choices for the first step and at most 3 choices for any later step. This implies that $\mathbb{P}_p(A_n) \leq \frac{4}{3}(3p)^n$ which goes to 0 as $n \to \infty$, since $p < \frac{1}{3}$. As $A_n \subseteq \{|C(\mathbf{0})| = \infty\}$ for all $n$ (compactness argument, see exercise 4), we have that $\Theta(p) = \mathbb{P}_p(|C(\mathbf{0})| = \infty) = 0$. □

The method of proof for the other end of the $p$ spectrum is often called a contour or Peierls argument, the latter named after the person who proved

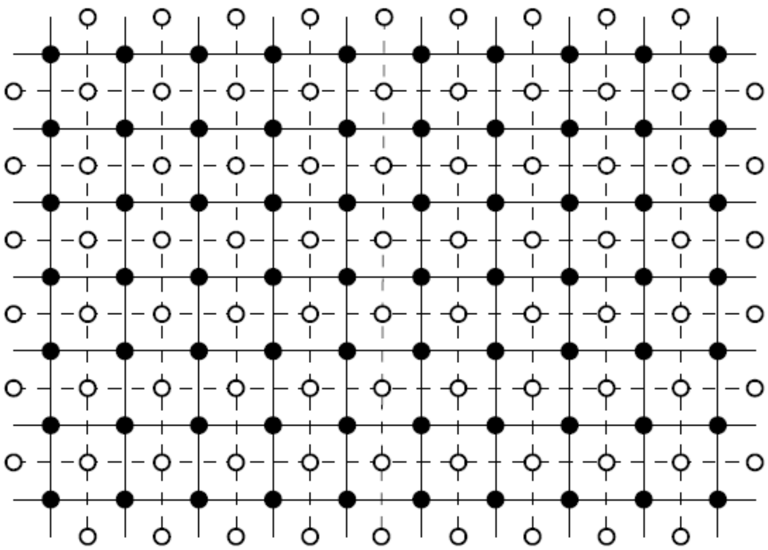

Figure 2: The dual graph of the square lattice is again the square lattice..

a phase transition for another model in statistical mechanics called the Ising model (see later). The first key ingredient is to introduce the so-called dual graph $(\mathbb{Z}^2)^*$ which is simply $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, i.e. the ordinary two-dimensional lattice translated by one half in each direction. Apparently, there is a 1-1 correspondence between the edges of $\mathbb{Z}^2$ and those of $(\mathbb{Z}^2)^*$, namely the pairs intersecting. Given a realization of open and closed edges of $\mathbb{Z}^2$, we obtain a similar realization for the edges of $(\mathbb{Z}^2)^*$ by simply calling an edge in the dual graph open if and only if the corresponding edge in $\mathbb{Z}^2$ is open. Observe that if the collection of open edges of $\mathbb{Z}^2$ is chosen according to $\mathbb{P}_p$ (as it is), then the distribution of the set of open edges for $(\mathbb{Z}^2)^*$ will trivially also be given by $\mathbb{P}_p$. A next key step is a result due to Whitney which is pure graph theory, namely that components in a graph correspond to closed circles in the dual graph, see exercise 5.

**Proposition 1.3**
*For bond percolation on $\mathbb{Z}^2$ it holds $\Theta(p) > 0$ for $p > \frac{2}{3}$.*

Proof: Let $B_n$ be the event that there is a simple cycle in $(\mathbb{Z}^2)^*$ surrounding $\mathbf{0}$ having length $n$, all of whose edges are closed. A similar counting argument (see exercise 6) bounds the numer of such cycles from above by $4n \cdot 3^{n-1}$. The probability that all edges of such a cycle are closed is $(1-p)^n$, by duality and independence. Hence we have

$$1-\Theta(p) = \mathbb{P}_p(|C(\mathbf{0})|<\infty) = \bigcup_{n=4}^{\infty} B_n \le \sum_{n=4}^{\infty} \mathbb{P}_p(B_n) \le \sum_{n=4}^{\infty} 4n\cdot 3^{n-1}(1-p)^n$$
$$\le 4\,(1-p)\sum_{n=1}^{\infty} n\cdot \big(3(1-p)\big)^{n-1},$$

which is less than 1 as long as $p>\frac{8}{9}$.

Using local modification, we can extend this result all the way down to $p>\frac{2}{3}$: Let $N$ be chosen so that $\sum_{n\ge N}^{\infty} 4n\cdot 3^{n-1}(1-p)^n<1$. Let $E_1$ be the event that all edges are open in $[N,N]\times[N,N]$ and $E_2$ be the event that there are no simple cycles in the dual surrounding $[N,N]^2$ consisting of all closed edges. Then $E_1$ and $E_2$ have positive probability and are independent so that $\mathbb{P}_p(|C(\mathbf{0})|=\infty)\ge \mathbb{P}(E_1\cap E_2)>0$. □

**Remark**
Defining the critical parameter

$$p_{\mathrm{c}} = \sup\{p\in[0,1]:\ \Theta(p)=0\} = \inf\{p\in[0,1]:\ \Theta(p)>0\}$$

and combining the two propositions above, we found $p_{\mathrm{c}}(2)\in[\frac{1}{3},\frac{2}{3}]$. Furthermore, checking the proof of Prop. 1.3 once more, we showed in fact that $\Theta(p)$ goes to 1 as $p\to 1$.

In 1960, Harris [28] proved that $\Theta(\frac{1}{2})=0$ and the conjecture made at that point was that $p_{\mathrm{c}}(2)=\frac{1}{2}$. However, it took 20 more years before there was a proof and this was done by Kesten [38]:

**Theorem 1.4**
*The critical value for bond percolation on $\mathbb{Z}^2$ is $\frac{1}{2}$.*

Without further complications we can conclude that the critical value $p_{\mathrm{c}}(d)$ is non-trivial even in dimensions greater than 2, see exercise 9.

## 1.4 Percolation function and uniqueness of the infinite cluster

For general dimension $d$ one usually writes $\Theta_d(p)$ for the probability that the origin percolates in the i.i.d. bond percolation process on $\mathbb{Z}^d$. From the previous section we know that $\Theta_1(p) = \mathbb{1}_{\{p=1\}}$, which is rather uninteresting, and that $\Theta_2(\frac{1}{2}) = 0$, while $\Theta_2(p) > 0$ for all $p > \frac{1}{2}$. Also we argued that $\Theta_{d+1}(p) \geq \Theta_d(p)$.

In fact, the *percolation function* $\Theta_d(p)$ is not only non-decreasing in $p$ but also continuous in the supercritical regime (i.e. where it is positive):

**Proposition 1.5**
*$\Theta_d(p)$ is a right-continuous function of $p$ on $[0, 1]$.*

PROOF: Let us write $f_n(p)$ for the probability that there is a self-avoiding path of open edges of length $n$ starting from the origin in the bond percolation on $\mathbb{Z}^d$. $f_n(p)$ is a polynomial in $p$ and by continuity from above and exercise 4 (with general dimension $d$ in place of $d = 2$), we have $f_n(p) \searrow \Theta_d(p)$ as $n \to \infty$. Now a decreasing limit of continuous functions is always upper semi-continuous and a non-decreasing upper semi-continuous function is right-continuous. $\square$

A much more difficult and deeper result is the left-continuity (due to van den Berg and Keane [6]), giving the above mentioned central result:

**Theorem 1.6**
*$\Theta_d(p)$ is continuous on $(p_c(d), 1]$.*

The proof of this uses another central result, namely the fact that the infinite cluster (if it exists) is unique, in other words there can only be one infinite cluster in i.i.d. bond percolation on $\mathbb{Z}^d$.

**Theorem 1.7**
*In supercritical bond percolation on $\mathbb{Z}^d$, the infinite cluster is unique.*

This is in fact true for all transitive graphs (with essentially the same proof), but for convenience, we stick to $\mathbb{Z}^d$. The main tools in the elegant proof due to Burton and Keane [8] are Birkhoff's ergodic theorem, local modification and the concept of trifurcation points.

The multivariate version of Birkhoff's theorem, attributed to Zygmund (see e.g. Thm. 10.12 in [37]), tells us that in an ergodic setting, spatial averages converge to the probabilistic average. For ease of notation, we write $\Lambda_n$ for the box $[-n,n]^d$ of side length $2n$ centered at $\mathbf{0}$ and $A(v)$ for a given event centered at vertex $v \in \mathbb{Z}^d$.

**Theorem 1.8**
*For an ergodic process (such as bond percolation) on $\mathbb{Z}^d$, it holds*

$$\lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} \mathbb{1}_{A(v)} = \mathbb{P}(A) \text{ almost surely.}$$

This immediately tells us that translation invariant events $A$ such as "there exist exactly 7 infinite clusters" have probability either 0 or 1. Using local modification we arrive at the following lemma due to Newman and Schulman:

**Lemma 1.9**
*The number of infinite clusters is either 0,1 or $\infty$.*

To rule out the possibility of infinitely many infinite clusters (in the supercritical setting) on $\mathbb{Z}^d$ (on other transitive graphs such as regular trees, this does occur!), we make use of the fact that our graph is *amenable*, which essentially means that volumes grow faster than surfaces for balls/boxes, via so-called *trifurcation points*: Let us call the node $v$ a trifurcation if

(a) $v$ is contained in an infinite cluster

(b) $C(v) \setminus \{v\}$ consists of exactly 3 infinite (and no finite) clusters.

**Lemma 1.10**
*If $\mathbb{P}_p$(there are infinitely many infinite clusters) $= 1$, as a consequence we have that $\mathbb{P}_p(\mathbf{0}$ is a trifurcation point$) > 0$.*

Proof: Let $A_n$ be the event that at least 3 infinite clusters intersect the box $\Lambda_n = [-n,n]^d$. If there are a.s. infinitely many infinite clusters, we can choose $n$ such that $\mathbb{P}_p(A_n)$ is strictly positive (by continuity from below).

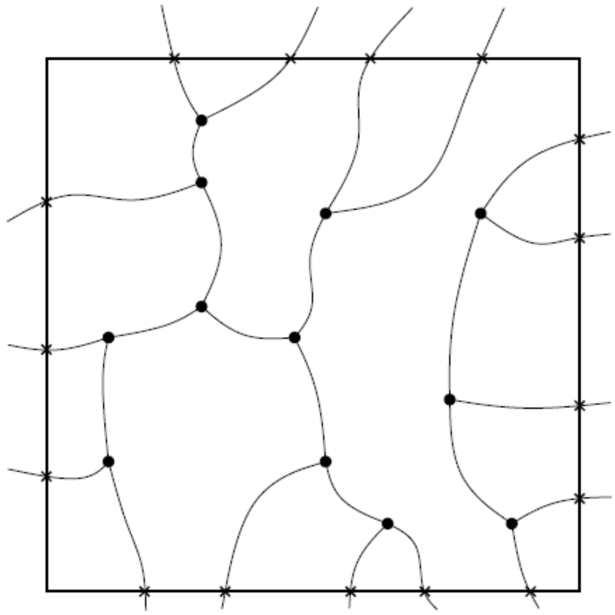

Figure 3: The set of trifurcation points in a box.

Now let us tweak this event a little and write $A'_n$ for the event that there are at least 3 infinite clusters, containing a vertex on the boundary of the box $\Lambda_n$. Then $A_n \subseteq A'_n$ and $A'_n$ is independent of the configuration inside the box. By local modification we can make $\mathbf{0}$ a trifurcation (with probability bounded away from 0). $\square$

To finish the proof of Theorem 1.7 one simply has to lead the implication of Lemma 1.10 to a contradiction with the amenability of $\mathbb{Z}^d$: Draw all trifurcations together with three infinite (disjoint) paths that originate from each of them (and might contain further trifurcations, see Fig. 3). While the number of trifurcations would have a positive density (Lemma 1.10, together with the translation invariance of percolation on $\mathbb{Z}^d$ and Birkhoff's Thm.), hence their number in $\Lambda_n$ would grow linearly in the volume (i.e. like $n^d$), the points on the boundary of the box on the paths cannot (as the boundary grows like $n^{d-1}$).

In the following proof of Theorem 1.6 (continuity of the percolation function) the uniqueness of the infinite cluster will come in at a crucial point:

Proof: We use the standard coupling of bond percolation on $\mathbb{Z}^d$ for different $p$, which gives that $\{e \text{ is } p_1\text{-open}\} \subseteq \{e \text{ is } p_2\text{-open}\}$ for all $p_1 \le p_2$.

Now let $C_p$ be the $p$-open cluster of the origin. In the standard coupling, obviously $C_{p_1} \subseteq C_{p_2}$ if $p_1 < p_2$ and $\Theta(p) = \mathbb{P}(|C_p| = \infty)$. Next, note that

$$\lim_{p' \nearrow p} \Theta(p') = \lim_{p' \nearrow p} \mathbb{P}(|C_{p'}| = \infty) = \mathbb{P}(|C_{p'}| = \infty \text{ for some } p' < p).$$

The last equality follows from using countable additivity in our big probability space (one can take $p'$ going to $p$ along a sequence). Due to the fact that $\{|C_{p'}| = \infty \text{ for some } p' < p\} \subseteq \{|C_p| = \infty\}$, what we need to show is

$$\mathbb{P}(\{|C_p| = \infty\} \setminus \{|C_{p'}| < \infty \text{ for all } p' < p\}) = 0.$$

Let $\gamma$ be such that $p_c(d) < \gamma < p$. Then almost surely there exists an infinite $\gamma$-open cluster $C$ (not necessarily containing the origin). Now, if $|C_p| = \infty$, then, by uniqueness applied to the $p$-open edges, we have that $C \subseteq C_p$ a.s. If $\mathbf{0} \in C$, we are of course done with $p' = \gamma$. Otherwise, there is a $p$-open path $\Gamma$ from the origin to $C$. Let $X = \max\{U_e : e \in \Gamma\}$ which is a.s. strictly smaller than $p$. So as soon as $p'$ is such that $X, \gamma < p' < p$, we have that there is a $p'$-open path from $\mathbf{0}$ to $C$ and therefore $|C_{p'}| = \infty$ as desired. $\square$

Combining the above results, we know that the percolation function can only have a jump in the critical point $p_c(d)$. In fact, for the square lattice, Harris' result shows it doesn't. Hence $\Theta_2(p)$ qualitatively looks like depicted below:

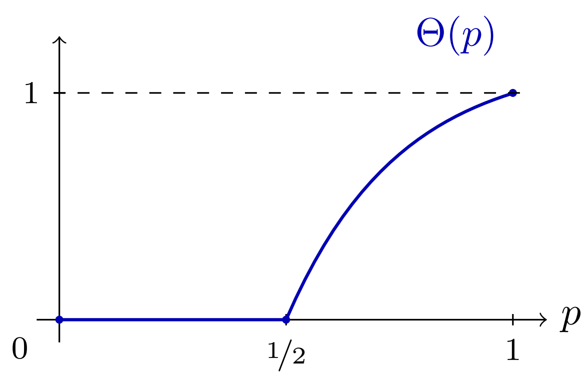


Figure 4: The percolation function for the square lattice.

Nobody expects to ever know what $p_c(d)$ is for $d \geq 3$ but a more interesting question to ask is what happens at the critical value itself, i.e. is $\Theta_d(p_c(d))$ equal to 0 or is it positive (hence the percolation function not continuous). The common belief is that $\Theta_d$ does not have a jump irrespectively of dimension. Interestingly, besides $d = 2$, this is also known to be the case for $d \geq 19$ (a highly nontrivial result by Hara and Slade [27] using a technique called *lace expansion*). Rather recently the dimensions 11 to 18 have been solved with similar methods but for $3 \leq d \leq 10$ it is not known and viewed as one of the major open questions in the field.

Variants of bond percolation that received ample attention are the aforementioned site percolation and a non-static version called *dynamical percolation*, in which the status of each edge (open/closed) is updated independently when a random Exp(1) time associated with the corresponding edge has elapsed.

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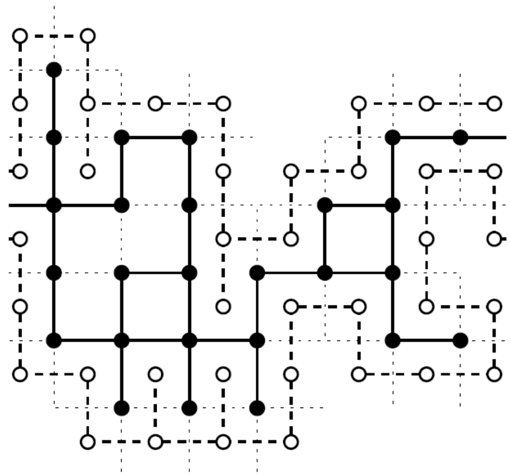
**Exercise 1**
Describe how to couple the binomial distributions $\text{Bin}(n, p)$ and $\text{Bin}(m, q)$, where $n \leq m$ and $p \leq q$ in a monotone way, i.e. define random variables $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, q)$ s.t. $\mathbb{P}(X \leq Y) = 1$. Is such a monotone coupling possible if one of the two conditions ($n \leq m,\ p \leq q$) is dropped?

**Exercise 2 (Doeblin's maximal coupling)**
Describe a way to maximally couple two discrete distributions $\mu, \nu$ in the sense that the pair $(X, Y)$, where $X \sim \mu$ and $Y \sim \nu$, fulfills $\mathbb{P}(X = Y) \geq \mathbb{P}(X' = Y')$ for all possible couplings $X' \sim \mu,\ Y' \sim \nu$.

**Exercise 3**
Show that $\Theta(p) = 1$ is impossible for all $p < 1$.

**Exercise 4**
For any subgraph of $\mathbb{Z}^2$ (hence any realization of bond percolation), show that $|C(\mathbf{0})| = \infty$ if and only if there is a self-avoiding path from 0 to $\infty$ consisting of open edges (i.e. containing infinitely many edges).

**Exercise 5**
Verify that the component containing the origin is finite if and only if there exists a simple cycle in $(\mathbb{Z}^2)^*$ surrounding $\mathbf{0}$, consisting of all closed edges. The following picture (from [25]) might inspire your ideas:



**Exercise 6**
Show that the number of cycles in $\mathbb{Z}^2$ around the origin of length $n$ is at most $4n \cdot 3^{n-1}$.

**Exercise 7**
Using Kolmogorov's 0-1 law (which says that all *tail events* have probability 0 or 1), show that $\mathbb{P}_p$(there exists an infinite component) is either 0 or 1.

(If you are unfamiliar with Kolmogorov's 0-1 law, one should say that there are many (relatively easy) theorems in probability theory which guarantee, under certain circumstances, that a given type of event must have a probability which is either 0 or 1. But they don't tell which of 0 and 1 it is, which is funnily enough almost always the harder thing to show).

**Exercise 8**
In the proof of Prop. 1.5: If you don't know what words like upper semicontinuous mean (and even if you do), redo the second part of the proof with your hands, not using anything.

**Exercise 9**
Show that the phase transition for bond percolation on $\mathbb{Z}^d$ is non-trivial, in the sense that $p_c(d) \in (0, 1)$, for all $d \geq 2$: First, mimick the proof of Prop. 1.2 to show that $p_c(d) \geq \frac{1}{2d-1}$, i.e. generalize the statement $p_c(2) \geq \frac{1}{3}$ to general dimension $d$. Then: How did we already prove $p_c(d) \leq \frac{2}{3}$?

# 2 First-passage percolation, branching random walk and frogs

A model closely related to that of bond percolation – and more suitable to be taken as a simplification of liquid permeating porous material – is that of *first-passage percolation* (FPP): To begin with, all edges are closed and there are i.i.d. waiting times associated with each edge. At time $t = 0$, only the origin $\mathbf{0}$ is considered to be active (wet). Once one of the endpoints of an edge $e = \langle u, v \rangle$, say $u$, becomes active, the clock starts ticking and once the waiting time attached to $e$ has elapsed, the edge is declared open and the other endpoint $v$ becomes active (if it isn't already).

The other two models introduced in this section – the *branching random walk* (BRW) and the *frog model* – are interacting particle systems ([43] is an extensive standard reference for this field): In BRW particles reproduce by giving rise to offspring according to a given distribution. The offspring is placed around the site of the mother particle independently and with the same (local) law. Similar to the growth of the "wet region" in FPP one can track the set of sites visited by particles and analyze its growth and shape.

The frog model operates somewhat similarly to BRW, however, there is no reproduction component: At the beginning sleeping frogs are placed at the sites of the lattice (in the simplest version one per site, but most work deals with general i.i.d. initial configurations with at least one frog at the origin). At the start, only all frogs located at the origin are considered active. Active frogs perform (independent) simple random walks on the lattice, waking up all frogs at sites they visit.

At first glance, there is no phase transition to be expected in these models (as the only parameters potentially inducing one involved here are in the distribution of the waiting time (FPP), the offspring and displacement distribution (BRW) and the initial configuration of frogs respectively). But once we consider a competition version of these models, this question becomes interesting (more details see the seminar talks next week).

## 2.1 First-passage percolation

Again, we will restrict our attention for this stochastic growth model to the lattice case, i.e. the graph $\mathbb{Z}^d$. This model has been extensively studied in the literature, and was introduced by Hammersley and Welsh in 1965. While it can be defined (and analyzed) for different (non-negative) distributions for the waiting times, the standard model assumes i.i.d. waiting times that are exponentially distributed with parameter 1 (for good arguments why, see exercise 10).

Let again $E$ denote the set of edges of the nearest-neighbor lattice $\mathbb{Z}^d$ and $(\tau_e)_{e\in E}$ denote a collection of i.i.d. random variabels, referred to as edge passage times. Define the passage time of a path $\Gamma$ as $T(\Gamma) := \sum_{e\in\Gamma} \tau_e$. In particular, one is interested in the travel time, also referred to as *first-passage time*, between two vertices $x$ and $y$ in $\mathbb{Z}^d$, which is defined as

$$T(x, y) := \inf\{T(\Gamma) : \ \Gamma \text{ is a path from } x \text{ to } y\}.$$

As mentioned before, first-passage percolation is often motivated as a model for the spatial propagation of a fluid when injected at the origin of the lattice. The term passage time reflects the interpretation of the random variables as the time needed for a fluid to traverse the edge. Similarly, first-passage times (between two points) are commonly interpreted as the time it would take a fluid injected at one point to reach another. Relevant questions aim to understand the spatial growth of the fluid injected at the origin of the lattice. How far will the fluid reach in a fixed time interval? How does the number of wet sites grow in time? What can be said about the shape of the set of wet vertices? All these questions concern the central object defined as

$$W_t := \{\mathbf{z} \in \mathbb{Z}^d : \ T(\mathbf{0}, \mathbf{z}) \leq t\}, \ \text{for } t \geq 0,$$

which can be interpreted as the wet region at time $t$. Investigating how first-passage times behave is essential in order to understand how the wet region evolves in time. However, the known picture is still far from complete. A survey by Kesten on the developments in first-passage percolation can be found in [39]; a more recent reference is the survey by Howard [35].

Particular efforts have been invested in studying the propagation of the fluid in coordinate directions. If $\mathbf{e}_1 = (1, 0, \ldots, 0)$ denotes the unit vector along the first coordinate axis of $\mathbb{Z}^d$, this corresponds to studying the (random) sequence $(T(\mathbf{0}, n\mathbf{e}_1))_{n\in\mathbb{N}}$. Basic questions about first-passage times were studied already in Hammersley and Welsh [26]. Under which conditions, and in which sense does $T(\mathbf{0}, n\mathbf{e}_1)/n$ converge as $n \to \infty$? Is the expected travel time $\mathbb{E}[T(\mathbf{0}, n\mathbf{e}_1)]$ increasing in $n$? First-passage times have a complex dependence structure, however, as defined they are easily seen to be subadditive, i.e.,

$$T(\mathbf{x}, \mathbf{y}) \leq T(\mathbf{x}, \mathbf{z}) + T(\mathbf{z}, \mathbf{y}), \quad \text{for any } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{Z}^d.$$

Since the distribution of $T(\mathbf{x}, \mathbf{x}+\mathbf{y})$ does not depend on the site $\mathbf{x} \in \mathbb{Z}^d$ it follows immediately from Fekete's lemma (see exercise 13) that the following limit exists:

$$\mu_{\mathbf{e}_1} := \lim_{n\to\infty} \frac{\mathbb{E}[T(\mathbf{0}, n\mathbf{e}_1)]}{n}.$$

While already Hammersley and Welsh proved in 1965 that $\limsup \frac{1}{n} T(\mathbf{0}, n\mathbf{e}_1)$ converges almost surely, applying Kingman's Subadditive Ergodic Theorem (a handy version of which can be found for example in [42], see below) readily gives a much stronger statement.

**Theorem 2.1 (Subadditive Ergodic Theorem)**
*Let $\{X_{m,n}\}_{0\leq m<n}$ be a collection of random variables satisfying the following four conditions:*

*(a)* $X_{0,n} \leq X_{0,m} + X_{m,n}$ *for all* $0 < m < n$.

*(b) The distribution of the sequence* $\{X_{m,m+k}\}_{k\in\mathbb{N}}$ *does not depend on* $m \geq 0$.

*(c) The sequence* $\{X_{km,(k+1)m}\}_{k\in\mathbb{N}}$ *is stationary for each* $m \geq 0$.

*(d) For all* $n \in \mathbb{N}$, $\mathbb{E}[|X_{0,n}|] < \infty$ *and* $\mathbb{E}[X_{0,n}] \geq -cn$ *for some constant* $c \in \mathbb{R}$.

*Then the following conclusions hold:*

*(i) The limit* $\gamma := \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[X_{0,n}]$ *exists and is equal to* $\inf_{n\in\mathbb{N}} \frac{1}{n}\mathbb{E}[X_{0,n}]$.

*(ii)* $X := \lim_{n\to\infty} \frac{1}{n} X_{0,n}$ *exists almost surely and in* $L^1$, *where* $\mathbb{E}[X] = \gamma$.

*Moreover, if all sequences in (c) are ergodic, then* $X = \gamma$ *almost surely.*

Applying this to the set of random variables $\{T_{m\mathbf{e}_1,n\mathbf{e}_1}\}_{0\leq m<n}$, we obtain the following

**Proposition 2.2**
*For FPP on $\mathbb{Z}^d$ with $L^1$ waiting times, it holds*

$$\lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n} = \mu_{\mathbf{e}_1} \quad \textit{almost surely and in } L^1.$$

Proof: Conditions (a), (b) and (c) in the Subadditive Ergodic Theorem are easily verified from the subadditivity and the translation invariance of the underlying i.i.d. structure of the lattice. To verify (d), it suffices to show $\mathbb{E}[T(\mathbf{0}, \mathbf{e}_1)] < \infty$, since $0 \le \mathbb{E}[T(\mathbf{0}, n\mathbf{e}_1)] \le n\,\mathbb{E}[T(\mathbf{0}, \mathbf{e}_1)]$ by the non-negativity and subadditivity of passage times. This, however, is immediate as $T(\mathbf{0}, \mathbf{e}_1)$ is dominated by the edge passage time $\tau_{\mathbf{e}_1}$. For more heavy-tailed edge passage times (that are not $L^1$), slightly more work and a different (weaker) moment condition are needed. Hence, the conditions of the Subadditive Ergodic Theorem are satisfied, and the limit $\lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n}$ exists almost surely and in $L^1$.

It remains to show that the limit is constant. Let again $\Lambda_n = [-n, n]^d$ denote the box of side length $2n$ centered at $\mathbf{0}$. With a slight abuse of notation, let $T(\Lambda_m, n\mathbf{e}_1) := \min_{v\in\Lambda_m} T(v, n\mathbf{e}_1)$. Obviously,

$$T(\Lambda_m, n\mathbf{e}_1) \le T(\mathbf{0}, n\mathbf{e}_1) \le T(\Lambda_m, n\mathbf{e}_1) + \sum_{e\in\Lambda_m} \tau_e.$$

So we can conclude for every $m \ge 0$ that

$$\lim_{n\to\infty} \frac{T(\Lambda_m, n\mathbf{e}_1)}{n} = \lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n} \quad \text{almost surely and in } L^1.$$

However, since $T(\Lambda_m, n\mathbf{e}_1)$ is independent of $\tau_e$ for all $e \in \Lambda_m$, the limit $\lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{e}_1)}{n}$ has to be as well. Since this holds for all $m \ge 0$, Kolmogorov's 0-1 law implies that it has to be constant (as the limit being $\ge c$ constitutes a tail event). $\square$

From Proposition 2.2 we obtain the propagation of the fluid in coordinate directions. Similarly, the Subadditive Ergodic Theorem applies to the sequence $\{T_{\mathbf{0},n\mathbf{z}}\}_{n\in\mathbb{N}}$ for any $\mathbf{z} \in \mathbb{Z}^d$. For practical purposes it is handy to extend the definition of passage times between vertices on $\mathbb{Z}^d$ to pairs of points in $\mathbb{R}^d$: For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, define $T(\mathbf{x}, \mathbf{y}) := T(\mathbf{x}^*, \mathbf{y}^*)$ where $\mathbf{x}^*$ and $\mathbf{y}^*$ denote the points in $\mathbb{Z}^d$ closest to $\mathbf{x}$ and $\mathbf{y}$, respectively (choosing the points closest to the origin in case of a tie, say). Then it is in fact possible (although not immediate from the Subadditive Ergodic Theorem) to show that for any $\mathbf{x} \in \mathbb{R}^d$ there exists the limit

$$\mu_{\mathbf{x}} := \lim_{n\to\infty} \frac{T(\mathbf{0}, n\mathbf{x})}{n} \quad \text{almost surely and in } L^1.$$

This limit is often referred to as *time constant*. Trivially, in one dimension, the time constant is just the expected value of the edge passage time (see exercise 12). Not much is known about the exakt values of $\mu$ in higher dimensions (not even in the coordinate axis direction). However, a lower bound $\mu_{\mathbf{e}_1} \geq 0.298$ on the square lattice $\mathbb{Z}^2$ for Exp(1) edge passage times was derived by Janson [36] already in 1981.

Hammersley and Welsh further conjectured that the expected travel time $\mathbb{E}[T(\mathbf{0}, n\mathbf{e}_1)]$ is non-decreasing in $n$ (for general edge passage time distributions). Despite the intuitive appeal the conjecture has, van den Berg [5] constructed an example that essentially shows that the conjecture is false for small $n$. It remains an open problem to find out whether the expected travel time could be monotonic for sufficiently large $n$.

## 2.2 The shape theorem

The convergence to the time constant describes the spatial growth of the process in any fixed direction. To understand the growth of the wet region, this convergence needs to be concluded in all directions simultaneously. This can be obtained, and was first realized by Cox and Durrett [13] for general waiting time distributions. In terms of first-passage times, their result can be stated as follows:

$$\limsup_{\mathbf{z}\in\mathbb{Z}^d} \frac{|T(\mathbf{0}, \mathbf{z}) - \mu_{\mathbf{z}}|}{\|\mathbf{z}\|} = 0 \quad \text{almost surely.}$$

Equivalently, this result can be stated in terms of the wet region. Just as first-passage times were extended to pairs of points in $\mathbb{R}^d$, it is convenient to replace $W_t$, which was defined as a subset of $\mathbb{Z}^d$, with a corresponding subset of $\mathbb{R}^d$. For $t \geq 0$, let $\mathcal{W}_t := \{\mathbf{x} \in \mathbb{R}^d : \ T(\mathbf{0}, \mathbf{x}) \leq t\}$. The above result can then be described as how closely $\mathcal{W}_t$ resembles the set

$$\mathcal{W} := \{\mathbf{x} \in \mathbb{R}^d : \ \mu_{\mathbf{x}} \leq 1\}.$$

often referred to as *asymptotic shape*.

Formulated in terms of the wet region, the result by Cox and Durrett is known as the *Shape Theorem*:

**Theorem 2.3**
*Consider first-passage percolation on $\mathbb{Z}^d$ with i.i.d. passage times (fulfilling a weak moment condition). If $\mu_{\mathbf{e_1}} > 0$, then for all $\varepsilon > 0$ almost surely*

$$(1-\varepsilon)\mathcal{W} \subset \tfrac{1}{t}\mathcal{W}_t \subset (1+\varepsilon)\mathcal{W}, \quad \textit{for } t \textit{ large enough.}$$

The geometric properties of the set $\mathcal{W}$ can in fact be divided into two regimes:

**Proposition 2.4**

*(i) $\mathcal{W}$ is compact, convex and has a non-empty interior when $\mu_{\mathbf{e_1}} > 0$*

*(ii) $\mathcal{W} = \mathbb{R}^d$, when $\mu_{\mathbf{e_1}} = 0$.*

In the first regime the Shape Theorem states that the wet region grows with linear speed. Except for convexity (and apparent symmetries with respect to coordinate axis), it has turned out very hard to prove further characteristics of $\mathcal{W}$ in this regime. The second case in turn translates to $K \subset \frac{1}{t}\mathcal{W}_t$ almost surely for arbitrary compact set $K \subset \mathbb{R}^d$ and $t$ large enough. When it comes to the characterizations of these two regimes, Kesten [39] showed that $\mu_{\mathbf{e_1}} = 0$ if and only if $\mathbb{P}(\tau_e = 0) \geq p_c(d)$, where $p_c(d)$ as before denotes the percolation threshold for bond percolation on the $\mathbb{Z}^d$ lattice.

To prove Prop. 2.4, one first has to establish appropriate properties of the time constant, namely:

(a) Linearity: $\mu_{a\mathbf{x}} = a\,\mu_{\mathbf{x}}$ for all $a \geq 0$ and $\mathbf{x} \in \mathbb{R}^d$

(b) Triangle inequality: $\mu_{\mathbf{x}+\mathbf{y}} \leq \mu_{\mathbf{x}} + \mu_{\mathbf{y}}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$

(c) Continuity: $\mu_{\mathbf{x}} - \mu_{\mathbf{y}} \leq d\,\mathbb{E}[T(\mathbf{0}, \mathbf{e}_1)] \cdot \|x - y\|$.

To verify these is rather straight forward: For $a \in \mathbb{N}$ it holds

$$\mu_{a\mathbf{x}} = a \lim_{n\to\infty} \frac{\mathbb{E}[T(\mathbf{0}, an\mathbf{x})]}{an} = a\,\mu_{\mathbf{x}}.$$

The extension to non-negative real $a$ is done via a comparison of $\mathbb{E}[T(\mathbf{0}, an\mathbf{x})]$ and $\mathbb{E}[T(\mathbf{0}, \lfloor an \rfloor \mathbf{x})]$, where $\lfloor . \rfloor$ denotes the integer part. The difference is easily seen to be bounded. For $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ the triangle inequality follows directly from the subadditivity and a similar comparison can be made to extend it to arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. For the continuity note that

$$\begin{aligned} |\mathbb{E}[T(\mathbf{0}, n\mathbf{x})] - \mathbb{E}[T(\mathbf{0}, n\mathbf{y})]| &\leq |\mathbb{E}[T(n\mathbf{x}, n\mathbf{y})]| \\ &\leq \mathbb{E}[T(\mathbf{0}, \mathbf{e}_1)] \, \|(n\mathbf{x})^* - (n\mathbf{y})^*\|_1 \\ &\leq d \, \mathbb{E}[T(\mathbf{0}, \mathbf{e}_1)] \, \|(n\mathbf{x})^* - (n\mathbf{y})^*\|. \end{aligned}$$

Dividing both sides by $n$ and then sending it to infinity shows the claimed continuity property.

With these in hand, we can show Prop. 2.4:

PROOF:

(i) The asymptotic shape $\mathcal{W}$ is convex in both regimes. To see this, note that $\mathbf{x}$ is contained in $\mathcal{W}$ if and only if $\mu_{\mathbf{x}} \leq 1$ by definition. Thus, if $\mathbf{x}$ and $\mathbf{y}$ belong to $\mathcal{W}$, and $\lambda \in (0, 1)$, then also $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$ belongs to $\mathcal{W}$ by linearity and triangle inequality. The remaining two properties of $\mathcal{W}$ when $\mu_{\mathbf{e_1}} > 0$ can be deduced with help of the convexity. First, note that by linearity, there are $a > 0$ and $b < \infty$ such that $\mu_{a\mathbf{e_1}} < 1$ and $\mu_{b\mathbf{e_1}} > 1$. Together with convexity and reflexion symmetry of $\mathcal{W}$, the former implies that $\mathcal{W}$ has non-empty interior, whereas the latter that $\mathcal{W}$ is bounded. To prove compactness, it remains to conclude that $\mathcal{W}$ is closed. However, that is immediate from the continuity property.

(ii) To conclude that $\mathcal{W} = \mathbb{R}^d$, when $\mu_{\mathbf{e_1}} = 0$, it suffices to prove that either $\mu_{\mathbf{x}} = 0$, for all $\mathbf{x}$ or $\mu_{\mathbf{x}} \neq 0$, for all $\mathbf{x} \neq \mathbf{0}$. Assume that the latter is not the case. First, assume that $\mu_{\mathbf{e_1}} = 0$, from which it follows that $\mu_{\mathbf{e_j}} = 0$ for each $j = 1, 2, \ldots, d$ by symmetry. That $\mu_{\mathbf{x}} = 0$, for all $\mathbf{x}$ is now immediate from linearity and triangle inequality. In general, if $\mu_{\mathbf{x}} = 0$, for some $\mathbf{x} \neq \mathbf{0}$, then we can in a similar fashion, via reflexion, obtain $d$ vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_d$ that span $\mathbb{R}^d$ and fulfill $\mu_{\mathbf{x}_j} = 0$ for each $j = 1, 2, \ldots, d$. Again, that $\mu_{\mathbf{x}} = 0$, for all $\mathbf{x}$ is immediate from linearity and triangle inequality. □

To show, that the Shape Theorem in fact is equivalent to the result by Cox and Durrett stated at the beginning of this section is not very deep, but another slightly technical matter.

## 2.3 Competition

Much later, Häggström and Pemantle [32] introduced the idea of two competing growth processes into FPP, a model that became known as two-type *Richardson model*. They considered a process on the square lattice, that starts at time 0 with an infection of type 1 at $\mathbf{0}$ and an infection of type 2 at another site $\mathbf{x}$, all other sites being healthy. The model can then be described in first passage percolation terms: a site $\mathbf{y}$ is infected at time $T(\{\mathbf{0}, \mathbf{x}\}, \mathbf{y})$, which we define as the infimum, over all paths starting at $\mathbf{0}$ or $\mathbf{x}$ and ending at $\mathbf{y}$, of the sum of the passage times along the path. Since the distribution of the passage times of the edges Exp(1) is continuous, it is not hard to see that the infimum is in fact a.s. a minimum which is attained for a unique path. If this fastest path starts at $\mathbf{0}$, then $\mathbf{y}$ gets infection of type 1, otherwise it gets type 2. One may think of the two-type Richardson model as a crude model for two growing bacterial colonies (or two political empires) competing for space. It may happen that at some early stage, one of the types of infection completely surrounds the other type which then is prevented from growing indefinitely (see exercise 14).

Write $A$ for the event that this does not happen, in which case both types of infection will grow indefinitely. The first question one would like to answer about the two-type Richardson model is whether or not $\mathbb{P}(A) > 0$ (it is obvious from exercise 14 that $\mathbb{P}(A) < 1$).

Their first central result shows that the answer to this question does in fact not dependent on $\mathbf{x}$:

**Proposition 2.5**
*With $\mathbb{P}_{\mathbf{0},\mathbf{x}}(A)$ denoting the probability of $A$ with the starting positions $\mathbf{0}$ and $\mathbf{x}$ respectively, it holds for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z}^2$:*

$$\mathbb{P}_{\mathbf{0},\mathbf{x}_1}(A) > 0 \iff \mathbb{P}_{\mathbf{0},\mathbf{x}_2}(A) > 0.$$

Intuitively, the least favorable starting position for coexistence, in the sense that both types occupy infinitely many sites, are neighboring initial seeds. In this respect, the main result of Häggström and Pemantle can be seen as an extreme-case analysis:

**Theorem 2.6**
*With $\mu$ denoting the time constant along the coordinate axes on $\mathbb{Z}^2$, it holds:*

$$\mathbb{P}_{\mathbf{0},\mathbf{e}_1}(A) > \tfrac{4\mu-1}{3}.$$

*Using the lower bound by Janson, this implies* $\mathbb{P}_{\mathbf{0},\mathbf{e}_1}(A) > 0.064$.

The proof of this theorem is based on a proposition stating that for any $\varepsilon > 0$ there exists some integer $m \geq 0$ so that

$$\limsup_{n\to\infty} \mathbb{P}\big[T\big(\mathbf{0},(n,m)\big) > T\big(\mathbf{0},(n-1,m)\big)\big] \geq \tfrac{2\mu+1}{3} - \varepsilon,$$

the latter being bigger than 0.532 (using Janson's bound). In words, this means that there are sites in $\mathbb{Z}^2$ arbitrarily far away from the origin which distinctly "feel" which direction the infection is coming from. While the proof of this proposition is rather technical, Prop. 2.5 can readily be settled using local modification:

Proof: Let us color type 1 vertices blue and type 2 vertices red. We can choose $t > 0$ and then $N$ big enough such that the set $W_t$ of all vertices colored at that time (eiter blue or red) contains a box of side length say $2\max\{\|\mathbf{x}_1\|, \|\mathbf{x}_2\|\}$ centered at the origin and is contained in a box of side length $N$ with probability bigger than $1-\mathbb{P}_{\mathbf{0},\mathbf{x}_1}(A)$. Note that the set of both blue and red vertices always form a (path-)connected set. By choice of $t$, the fact that $N$ is finite and since $\mathbb{P}_{\mathbf{0},\mathbf{x}_1}(A) > 0$ there exists a configuration on $W_t$ that has a positive probability conditioned on $A$. Keeping the configuration on the boundary of $W_t$ fixed, we can change the configuration inside $W_t$ such that $\mathbf{0}$ is still blue, both $\mathbf{x}_1$ and $\mathbf{x}_2$ are red, it coincides with the old configuration on the boundary on $W_t$ and the sets of both colors are still connected. Since this configuration has positive probability to occur at time $t$, when starting at sites $\mathbf{0}$ and $\mathbf{x}_2$ respectively, by the strong Markov property it follows that $\mathbb{P}_{\mathbf{0},\mathbf{x}_2}(A) > 0$ as well. $\square$

In a second paper two years later, the same authors proved a somewhat peculiar result for the same model but with the infections spreading at different rates (by scaling, one can be chosen to have passage times Exp(1), while the other has Exp($\lambda$) for some $\lambda \neq 1$). It states that for all but at most countably many values of $\lambda$ the event $A$ has probability 0. These potential exceptional values are due to a Fubini-type argument in the proof, but no one believes they actually exist. A few years later, their initial result (about possible coexistence) was generalized to higher dimensions (among others by Garet and Marchand [22], using proof techniques not dependent of the planar structure of $\mathbb{Z}^2$ or the bound on the time constant). The key idea in the proof of Garet and Marchand is to take starting positions $\mathbf{0}$ and $n\mathbf{e_1}$, for $n$ fixed large enough, and show that the passage time from $n\mathbf{e_1}$ to $-m\mathbf{e_1}$ is substantially larger than the passage time from $\mathbf{0}$ to $-m\mathbf{e_1}$ for large enough $m$. This can be used to show that given these starting positions both types will dominate on "their side"in the first coordinate direction. In view of Prop. 2.5 (generalized to higher dimensions) this is all that was needed. So the following conjecture is almost fully proved:

**Conjecture 2.7**
For the two-type Richardson model on $\mathbb{Z}^d$, $d \geq 2$, with infections spreading from $\mathbf{0}$ and $\mathbf{e_1}$ at rates 1 and $\lambda$ respectively, it holds

$$\mathbb{P}(A) > 0 \quad \text{if and only if } \lambda = 1.$$

Shortly after, Garet and Marchand also extended the result that two infections spreading with two different passage time distributions (one stochastically smaller than the other) cannot coexist, to distributions other than exponential [23]. In this more general setting they showed, however, not that the event $A$ (both occupy infinitely many sites; for distinction reasons sometimes called *weak coexistence*) has probability 0, but the smaller event that both types occupy at least a linear fraction of the sites (correspondingly called *strong coexistence*).

## 2.4 Frogs

The so-called frog model on $\mathbb{Z}^d$ is driven by moving particles (frogs) on the sites of the $d$-dimensional lattice. Each site $\mathbf{x} \in \mathbb{Z}^d$ is assigned an initial number $\eta(\mathbf{x})$ of particles, where $\{\eta(\mathbf{x})\}_{\mathbf{x}\in\mathbb{Z}^d}$ are i.i.d. Each particle is then independently equipped with a discrete time simple symmetric random walk. At time 0, the particles at the origin are activated, while all other particles are sleeping. When a particle is activated, it starts moving according to its associated random walk so that, in each time step, it moves to a uniformly chosen neighboring site. When a site is visited by an active particle, any sleeping particles at that site are activated and start moving (independently). If the origin is non-empty, this means that the set of activated particles grows to infinity.

The frog model has previously been studied, for example with respect to transience/recurrence [47], the shape of the set of visited sites [2, 3] and extinction/survival for a version of the model including death of active particles [1].

With $W_t$ denoting the set of sites visited by time $t$, or rather it continuous embedding, i.e.

$$\mathcal{W}_t = \left\{\mathbf{x} + (-\tfrac{1}{2}, \tfrac{1}{2}]^d : \mathbf{x} \text{ has been visited by an active particle at time } t\right\},$$

very similar to Theorem 2.3 (FPP), the Shape Theorem for the frog model reads as follows:

**Theorem 2.8**
*Consider the frog model on $\mathbb{Z}^d$ with the initial number of frogs being i.i.d. $\eta$. If $\eta(\mathbf{0}) > 0$, then there is a non-empty convex set $\mathcal{W}$ such that for all $\varepsilon > 0$ almost surely*

$$(1-\varepsilon)\mathcal{W} \subset \tfrac{1}{t}\mathcal{W}_t \subset (1+\varepsilon)\mathcal{W}, \quad \textit{for } t \textit{ large enough.}$$

In [2] Alves, Machado and Popov first proved this result for the original version of the model with deterministic starting configuration (one frog per site) and then generalized it together with Ravishankar to i.i.d. random initial numbers per site. The main obstacle in this generalization in fact posed the sites that are empty in the initial configuration.

When it comes to the model with death, i.e. where active particles vanish in each time step with probability $1-p$ the same set of authors found another interesting phase transition (in the parameter $p$) concerning the survival of the process as a whole:

**Theorem 2.9**
*For the frog model with death (survival probability $p < 1$) on $\mathbb{Z}^d$, it holds the following:*

*(i) If $\mathbb{E}\big(\log(\max\{\eta, 1\})^d\big) < \infty$, then $p_c(\eta) > 0$ and*

*(ii) for $d \geq 2$, $p_c(\eta) < 1$, if not $\eta \equiv 0$.*

Here $p_c(\eta)$ of course denotes the infimum of all $p$ for which the process survives with strictly positive probability. In one dimension, the process almost surely dies out for all $p < 1$ (given the weak moment condition

$\mathbb{E}\big(\log(\max\{\eta, 1\})\big) < \infty$), so that the phase transition one $\mathbb{Z}$ is again trivial in the sense that $p_c(\eta) = 1$.

Together with Mia Deijfen and Fabio Lopes, we introduced a two-type version of the model (without death), where an active particle can be of either of two types [17]. Activated particles are assigned the same type as the particle activating them (with an arbitrary tie breaker where necessary). We studied the possibility for the types to activate infinitely many particles and investigated in particular the event of (weak) coexistence, which is said to occur if both types activate infinitely many particles. Our results make heavy use of the shape theorem as well as the technique used by Garet and Marchand for FPP and in general resemble the results for competing FPP (for more details, see [17] or the seminar talk next week).

## 2.5 Branching Random Walk

As alluded to above, the model of BRW on $\mathbb{Z}^d$ proceeds (in discrete time) as follows: Given an offspring distribution $\eta$ and a local displacement distribution $\nu$, in each time step existing particles vanish and each of them independently gives rise to a random number of particles distributed according to $\eta$, which are independently placed (according to $\nu$) around the site where the mother particle vanished.

BRW has been a very active topic in contemporary probability the last two decades, see [45] for a survey covering mainly the one-dimensional case. BRW in higher dimensions is less well understood, but shape theorems (for the set of sites visited by particles) can be found in [7] and [12]. The model is well suited to describe spatial evolution of biological populations and versions of the model incorporating competition have been analyzed in this context.

The competition in these models, however, amounts to a single type of particles competing with each other in that there are constraints on the particle density or mass. An example of a two-type competition model was provided by Etheridge [20], in which the number of particles in bounded regions is limited. Together with Mia Deijfen, we introduced and analyzed a model, in which there are no limitations on the particle density, but competition arises in that the first type to reach a site is given a perpetual local advantage by claiming the site for its type. Interaction arises as particles landing on sites that are already claimed by the other type, adopt the type of the site (i.e. switch their type) with probability $p \in [0,1]$. Since there is no obvious monotonicity in $p$ (as this parameter governs the strength of interaction, hence affects both types simultaneously) this model is considerably harder to analyze, but we managed to derive at least partial results concerning the probability of coexistence (see [18] or the seminar talk next week).

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**Exercise 10**
A continuous random variable $X$ is called exponentially distributed with parameter $\lambda > 0$, if it has the density

$$f(x) = \lambda \exp(-\lambda x), \quad \text{for } x \geq 0.$$

Verify the following three properties:

(a) Scaling: If $X \sim \text{Exp}(\lambda)$, then $cX \sim \text{Exp}(\frac{\lambda}{c})$ for any $c > 0$.

(b) Minimum: If $X \sim \text{Exp}(\lambda_1)$ and $Y \sim \text{Exp}(\lambda_2)$ are independent, it holds

$$\min(X, Y) \sim \text{Exp}(\lambda_1 + \lambda_2).$$

(c) Memoryless property: The conditional distribution of $X \sim \text{Exp}(\lambda)$ given the event $\{X \geq c\}$ is again $\text{Exp}(\lambda)$, for any fixed $c > 0$.

*Hint*: The proofs of both (a) and (b) are very straight forward using the cdf.

**Exercise 11**
Show that the convolution of independent $\text{Exp}(\lambda)$ distributed random variables is gamma distributed, more precisely if $X_i \sim \text{Exp}(\lambda)$ are independent, then $X_1 + \ldots + X_n \sim \Gamma(n, \lambda)$, where a $\Gamma(n, \lambda)$ random variable has density

$$f(x) = \tfrac{\lambda^n}{(n-1)!}\, x^{n-1} \exp(-\lambda x), \quad \text{for } x \geq 0.$$

**Exercise 12**
Use the Subadditive Ergodic Theorem to conclude that the time constant for FPP on $\mathbb{Z}$ with i.i.d. $L^1$ edge passage times equals the expectation of this distribution. So for example if the waiting times are $\text{Exp}(\lambda)$, we get $\mu = \frac{1}{\lambda}$. Do you recognize that this is a rather unconventional way of reproving the strong law of large numbers (SLLN)?

**Exercise 13**
Prove *Fekete's Lemma*, which goes as follows:
Let $f : \mathbb{N} \to [0; \infty)$ be subadditive, i.e. such that for all $m, n \in \mathbb{N}$ it holds $f(m+n) \leq f(m) + f(n)$. Then $\lim \frac{f(n)}{n} = \inf \frac{f(n)}{n}$.

*Hint*: Fix $m \in \mathbb{N}$ and for $n \in \mathbb{N}$, write $n = mq + r$, where $0 \leq r < m$ (division with remainder), in order to deduce from subadditivity that

$$\tfrac{f(n)}{n} \leq \tfrac{q-1}{n} f(m) + \tfrac{1}{n} \max_{0 \leq i < m} f(m+i).$$

This implies $\limsup \frac{f(n)}{n} \leq \frac{f(m)}{m}$ and taking the infimum over $m$ completes the proof.

**Exercise 14**
Given that the two types of infections in the two-type Richardson model on $\mathbb{Z}^2$ start at sites $\mathbf{0}$ and a neighboring site and spread with Exp(1) edge passage times, calculate a lower bound on the probability that one will completely surround the other.

*Hint*: Exercises 10 and 11 might be helpful here. With little (integration) work you can show $\mathbb{P}_{\mathbf{0},\mathbf{e}_1}(A^c) \geq \frac{2}{125^2} = 0.000128$.

# 3 Magnets and opinion formation processes

## 3.1 Ising Model

Just like water changing its state of matter depending on the temperature, ferromagnetic material undergoes a phase transition in the sense that macroscopic properties of the matter are changed. Well above a certain critical temperature, the ferromagnetic material is unmagnetic on a macroscopic scale (if not exposed to a strong external magnetic field); well below this temperature however, a phenomenon that is called *spontaneous magnetization* occurs: the microscopic magnetic dipole moments, originating from atomic spins, start to align and turn the material into a magnet – even in the absence of an external field.

Already in 1907, Pierre Weiss tried to explain this behavior, building on earlier work by Pierre Curie. He used an approach that became known as *mean* *field theory*: In a large statistical system, the effects of all other particles on one fixed particle is replaced by their statistical average. This approximation turns a many-body problem with interactions, which in general is very difficult to solve exactly, into a one-body problem with external field. Clearly, this is a rather crude simplification as the fluctuating interaction of the considered particle with the rest of the system is approximated by a time-independent effective field. Nevertheless, it made the spin problem tractable and allowed Weiss to draw conclusions explaining the two different phases of ferromagnetic material. The mean field theory approximation is however only qualitatively accurate and fails to give satisfactory answers to questions about the behavior near the phase transition. For temperatures near the critical one, the actual local magnetic fields are rapidly varying in time and consequently turn their statistical average into a quite poor representation of their effect.

A slightly different approach to explain ferromagnetic behavior was the following theoretical model that physicist Wilhelm Lenz invented in 1920 and proposed to his student Ernst Ising for further studies two years later: A collection of atoms is arranged to form an atomic lattice represented by a graph $G = (V, E)$. Their elementary magnetic dipoles, often simply called *spins*, can be either in the state "up" or "down", represented by the numerical values $+1$ and $-1$ respectively. All spins taken together form what is called a spin configuration $\sigma \in \{-1, +1\}^V$. Neighboring spins that agree correspond to a lower *energy* than those that disagree. On a finite graph, the energy corresponding to a given configuration is simply calculated as twice the number of disagreeing neighboring pairs:

$$H(\sigma) = 2 \sum_{\langle u,v \rangle \in E} \mathbb{1}_{\{\sigma(u) \neq \sigma(v)\}}$$

Although the model can be defined on general graphs, we will consider the spins to be arranged on the lattice $\mathbb{Z}^d$. The one-dimensional Ising model was solved by Ising (1925) alone in his 1924 thesis, it has no phase transition. The two-dimensional Ising model on the square lattice $\mathbb{Z}^2$ is much harder (in his thesis Ising erroneously claimed that his results should generalize to higher dimensions) and was only given an analytic description much later, by Lars Onsager in 1944. It is, however, one of the simplest statistical models to show a phase transition. Though it is a highly simplified model of a magnetic material, the Ising model in three dimensions can still provide qualitative results applicable to real physical systems.

Let us first properly define the Ising model or rather the corresponding Gibbs measure on a finite graph:

**Definition 2**
For a finite graph $G = (V, E)$ and the *inverse temperature* $\beta > 0$, the *Gibbs measure* $\nu_G^\beta$ on $\{-1, +1\}^V$ is defined as the probability measure that assigns to each configuration $\sigma$ the probability

$$\nu_G^\beta(\sigma) = \tfrac{1}{Z_G^\beta}\, \mathrm{e}^{-\beta H(\sigma)},$$

where the *partition function* $Z_G^\beta = \sum_\sigma \mathrm{e}^{-\beta H(\sigma)}$ is just a normalizing constant.

To get started, think about what happens in the extreme cases, namely $\beta \to 0$ (corresponding to infinite temperature) and $\beta \to \infty$ (corresponding to the zero temperature limit), see exercise 15. If you are familiar with Markov chains, try to figure out the Gibbs measure on a finite path (exercise 16).

The Ising model on a general graph $G = (V, E)$ satisfies what is called the *Markov random field* property: Let $X$ be a random spin configuration on $G$ distributed according to $\nu_G^\beta$ and for $W \subseteq V$ let $\partial W$ denote the outer vertex boundary of $W$ (i.e. the set of nodes outside of $W$ that have a neighbor in $W$). Then the conditional distribution of $X(W)$ given $X(W^\mathrm{c})$ depends on $X(W^\mathrm{c})$ only via $X(\partial W)$. In this context, $W^\mathrm{c} := V \setminus W$ denotes the *complement* of the set $W$.

Before we turn to $\mathbb{Z}^d$, let us introduce another tool that is particularly useful when working with the Ising model and an elegant link between bond percolation and the former:

**Definition 3**
For a finite graph $G = (V, E)$ and parameters $p \in [0, 1],\ q > 0$, the *random-cluster measure* $\mu_G^{p,q}$ on $\{0, 1\}^E$ is given by

$$\mu_G^{p,q}(\eta) = \tfrac{1}{Z_G^{p,q}} \prod_{e \in E} p^{\eta(e)} (1-p)^{1-\eta(e)} q^{k(\eta)},$$

for every $\eta \in \{0, 1\}^E$, where $k(\eta)$ denotes the number of connected components in $(V, E_\eta)$ and $E_\eta$ denotes the subset of edges with $\eta(e) = 1$.

Note that $q = 1$ gives back the ordinary i.i.d. bond percolation on $G$. For $q = 2$ we can couple the Ising model and the random-cluster measure on $G$ in the following way:

(a) Assign the spins $+1$ and $-1$ to all vertices via fair independent coin flips.

(b) Independently of the spins, assign values 1 and 0 to the edges via independent biased coin flips (1 has probability $p$, 0 probability $1 - p$).

(c) Condition on the event that no two vertices with different spin are linked by an edge with value 1.

Then the projection of this conditional distribution on the edges equals the random-cluster measure $\mu_G^{p,2}$ and its projection onto the nodes (i.e. $\{-1, +1\}^V$ equals the Gibbs measure $\nu_G^\beta$ with $\beta = -\frac{1}{2}\log(1-p)$. (for a full proof, see Thm. 2.5 in [29], for intuition why this should be true, see exercise 17).

This coupling comes with a number of useful implications:

**Corollary 3.1**

*(i) First pick $Y \in \{0,1\}^E$ according to $\mu_G^{p,2}$ and then pick $X \in \{-1,1\}^V$ by assigning spins to the connected components by independent fair coin flips. Then $X$ has distribution $\nu_G^\beta$ with $\beta = -\frac{1}{2}\log(1-p)$.*

*(ii) First pick $X \in \{-1,1\}^V$ according to $\nu_G^\beta$ and then for each edge $e = \langle u, v\rangle \in E$ independently, set*

$$Y(e) = \begin{cases} 1 & \textit{with probability } \begin{cases} 1 - \mathrm{e}^{-2\beta} & \textit{if } X(u) = X(v) \\ 0 & \textit{otherwise} \end{cases} \\ \\ 0 & \textit{with probability } \begin{cases} \mathrm{e}^{-2\beta} & \textit{if } X(u) = X(v) \\ 1 & \textit{otherwise.} \end{cases} \end{cases}$$

*Then $Y \in \{0,1\}^E$ is distributed according to $\mu_G^{p,2}$ with $p = 1 - \mathrm{e}^{-2\beta}$.*

**Corollary 3.2 (Positive correlation)**

*For the Ising model on $G = (V, E)$ with $\beta > 0$, spins are positively correlated, i.e.*

$$\mathbb{E}[X(u)\,X(v)] \geq \mathbb{E}[X(u)] \cdot \mathbb{E}[X(v)].$$

Proof: Pick $X$ as in Cor. 3.1 (i). Then

$$\begin{aligned}\mathbb{E}[X(u)\,X(v)] &= \mathbb{P}\big(X(u)=X(v)\big) - \mathbb{P}\big(X(u)\neq X(v)\big)\\ &= 2\,\mathbb{P}\big(X(u)=X(v)\big) - 1 = 2\,\mathbb{P}\big(X(u)=X(v)\big) - 1\\ &= 2\,\mathbb{P}\big(u \overset{Y}{\rightsquigarrow} v\big) + 2\,\mathbb{P}\big(u \overset{Y}{\not\rightsquigarrow} v, X(u)=X(v)\big) - 1\\ &= \mathbb{P}\big(u \overset{Y}{\rightsquigarrow} v\big) \geq 0 = \mathbb{E}[X(u)]\cdot\mathbb{E}[X(v)]\end{aligned}$$

□

**Lemma 3.3**
*For $\mu_G^{p,q}$ any edge $e=\langle u,v\rangle\in E$ and any $\eta'\in\{0,1\}^{E\setminus\{e\}}$, it holds*

$$\mu_G^{p,q}\big(\eta(e)=1\,|\,\eta'\big) = \begin{cases} p & \text{if } u \overset{\eta'}{\rightsquigarrow} v\\ \frac{p}{p+(1-p)q} & \text{otherwise.}\end{cases}$$

Proof: To see this, it is easiest to look at the odds ratio:

$$\frac{\mu_G^{p,q}(e \text{ is open}\,|\,\eta')}{\mu_G^{p,q}(e \text{ is closed}\,|\,\eta')} = \begin{cases} \frac{p}{1-p} & \text{if } u \overset{\eta'}{\rightsquigarrow} v\\ \frac{p}{(1-p)q} & \text{if } u \overset{\eta'}{\not\rightsquigarrow} v.\end{cases}$$

□

Let $\mu \overset{d}{\preceq} \mu'$ denote *stochastic domination* of a measure $\mu'$ over $\mu$, which corresponds to the existence of two random objects $X\sim\mu$ and $Y\sim\mu'$ such that $\mathbb{P}(X\preceq Y)=1$.

**Lemma 3.4**
*For $q\geq 1$ and $p^* := \frac{p}{p+(1-p)q}$, it holds*

$$\mu_G^{p^*,1} \overset{d}{\preceq} \mu_G^{p,q} \overset{d}{\preceq} \mu_G^{p,1}$$

The easiest way to prove this is to pick $X_0\sim\mu_G^{p^*,1}$ and $Y_0\sim\mu_G^{p,q}$ and then update one edge at a time simultaneously for both by resampling it (this is called Gibbs sampler) with the appropriate probabilities and using a monotone coupling (and Lemma 3.3). Then $X_{|E|}\preceq Y_{|E|}$. In the same way, the second domination can be proved.

Using the same type of argument (see Thm. 3.5 in [29] for a detailed proof), one can prove Holley's Theorem:

**Theorem 3.5**
*For finite $V$ and finite state space $S \subseteq \mathbb{R}$, let $\mu$ and $\mu'$ be probability measures on $S^V$ with full support. and let $X \sim \mu$ and $X' \sim \mu'$ be the corresponding $S^V$-valued random objects. If for all $v \in V$, $s \in S$ and all $\eta, \xi \in S^{V\setminus\{v\}}$ with $\eta \preceq \xi$ it holds*

$$\mathbb{P}\big(X(v) \geq s \,|\, X(V \setminus \{v\} = \eta\big) \leq \mathbb{P}\big(X'(v) \geq s \,|\, X'(V \setminus \{v\} = \xi\big),$$

*then $\mu \overset{d}{\preceq} \mu'$.*

With all this in hand, we can move on and lift the definition of the Ising model/Gibbs measure to the (infinite) lattice:

**Definition 4**
A probability measure $\nu$ on $\{-1,+1\}^{\mathbb{Z}^d}$ is called a *Gibbs measure* for the Ising model with inverse temperature $\beta > 0$ on $\mathbb{Z}^d$ if for all finite $W \subset \mathbb{Z}^d$, all $\sigma' \in \{-1,+1\}^{\partial W}$ and all $\sigma \in \{-1,+1\}^W$ we have

$$\nu\Big(X(W) = \sigma \,|\, X(\partial W\Big) = \sigma')$$
$$= \frac{1}{Z^\beta_{\sigma'}} \exp\Big(-2\beta\,\big(\sum_{\substack{u\sim v\\ u,v\in W}} \mathbb{1}_{\{\sigma(u)\neq\sigma(v)\}} + \sum_{\substack{u\sim v\\ u\in W,\ v\in\partial W}} \mathbb{1}_{\{\sigma(u)\neq\sigma'(v)\}}\Big)$$

and $\nu$ satisfies the Markov random field property.

For the lattice (or infinite graphs in general) the question of existence and uniqueness of Gibbs measures becomes interesting. And it is here we find the phase transition alluded to earlier:

**Theorem 3.6**
*For the Ising model on $\mathbb{Z}^d$, with $d \geq 2$, there exists a critical value $\beta_c = \beta_c(d) \in (0,\infty)$ such that*

*(i) for $\beta < \beta_c$, there is a unique Gibbs measure and*

*(ii) for $\beta > \beta_c$, there exist multiple Gibbs measures.*

For an outline of its proof, we need two more lemmas: For finite $W \subset \mathbb{Z}^d$ and $\sigma' \in \{-1,+1\}^{\partial W}$ let us write $\nu^\beta_{W,\sigma'}$ for the distributon on $\{-1,+1\}^W$ prescribed in Definition 4.

**Lemma 3.7**
*If $\sigma'_1, \sigma'_2 \in \{-1,+1\}^{\partial W}$ are such that $\sigma'_1 \preceq \sigma'_2$, it holds*

$$\nu^{\beta}_{W,\sigma'_1} \stackrel{d}{\preceq} \nu^{\beta}_{W,\sigma'_2}.$$

To see this, remember that the conditional probability of a $+1$ spin at $v$ given all other spin values is given by $\frac{1}{1+\exp(2\beta(2d-2\kappa))}$, where $\kappa$ denotes the number of $+1$ neighbors of $v$. This is increasing in $\kappa$ and thus increasing in the full surroundings. Holley's theorem applies.

For a box $\Lambda_n = \{-n,\dots,n\}^d$ let $\nu^{\beta}_{n,+}$ denote the probability measure on $\{-1,+1\}^{\mathbb{Z}^d}$ obtained by letting everything outside $\Lambda_n$ take spin value $+1$ and according to $\nu^{\beta}_{\Lambda_n,\mathbf{1}}$ on $\Lambda_n$. An application of the above Lemma guarantees that the limiting distribution $\nu^{\beta}_{+} := \lim_{n\to\infty} \nu^{\beta}_{n,+}$ exists (by monotonicity). Note that $\nu^{\beta}_{n,+}$ is a particular Gibbs measure for the Ising model on $\mathbb{Z}^d$ called the *plus measure*. Similarly, we can define $\nu^{\beta}_{n,-}$.

Any Gibbs measure $\nu^{\beta}$ satisfies $\nu^{\beta}_{n,-} \stackrel{d}{\preceq} \nu^{\beta} \stackrel{d}{\preceq} \nu^{\beta}_{n,+}$, hence $\nu^{\beta}_{-} \stackrel{d}{\preceq} \nu^{\beta} \stackrel{d}{\preceq} \nu^{\beta}_{+}$ by taking limits (using Lemma 3.7 in both cases). This implies the following:

**Lemma 3.8**
*For the Ising model on $\mathbb{Z}^d$ with inverse temperature $\beta > 0$ the following are equivalent:*

*(a) There is a unique Gibbs measure.*

*(b)* $\nu^{\beta}_{-} = \nu^{\beta}_{+}$

*(c)* $\nu^{\beta}_{+}(+1 \text{ spin at } \mathbf{0}) = \frac{1}{2}$

*(d)* $\lim_{n\to\infty} \nu^{\beta}_{n,+}(+1 \text{ spin at } \mathbf{0}) = \frac{1}{2}$

*(e)* $\lim_{n\to\infty} \mu^{p,2}_{\Lambda^*_n}(\mathbf{0} \rightsquigarrow \partial\Lambda^*_n) = 0$*, where $\Lambda^*_n$ is the finite graph given by $\Lambda_n$ joined by a node representing all of the outer vertex boundary $\partial\Lambda_n$.*

To prove Thm. 3.6 we can now verify (e):

- If $\beta$ is small enough, so that $p = 1 - \mathrm{e}^{-2\beta} < p_{\mathrm{c}}(\mathbb{Z}^d)$, the limiting probability is 0 (Lemma 3.4).

- If $\beta$ is large enough, so that $\frac{p}{p+2(1-p)} > p_{\mathrm{c}}(\mathbb{Z}^d)$, the limiting probability is strictly positive (Lemma 3.4).
- It is also increasing in $p$ (hence in $\beta$) by taking the single-edge conditional probabilities from Lemma 3.4 and applying Holley's Theorem. This concludes the (outline of the) proof.

Note that any convex combination of $\nu_{-}^{\beta}$ and $\nu_{-}^{\beta}$ also is a Gibbs measure on $\mathbb{Z}^d$, hence there are infinitely many if these two do not coincide. That raises the question if there are yet others. Somewhat surprisingly: In two dimensions no, but for large enough $\beta$ in dimension $d \geq 3$ there are (called Dobrushin states, obtained as a limit with the outer configuration on $\Lambda_n$ being plus on the upper and minus on the lower half).

On the infinite $d$-dimensional grid $\mathbb{Z}^d$, we can consider the spatial average of spins which is called *magnetization* of the material and defined by

$$\langle \sigma \rangle = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \sum_{v\in\Lambda_n} \sigma_v,$$

where $\Lambda_n = \{-n, \ldots, n\}^d$. With this notion of an average spin, we can distinguish between a paramagnetic, disordered phase in which the magnetization is almost surely 0 and a ferromagnetic, ordered phase in which non-zero magnetization has positive probability.

As already mentioned, Ising analyzed in his PhD thesis the one-dimensional case and found that the correlation of spin values decays exponentially with the distance of two sites, which implies that the magnetization equals 0. He erroneously concluded that the model does not feature any phase transition even in higher dimensions. This claim was proven wrong by Rudolf Peierls about one decade later. He investigated the Ising model on $\mathbb{Z}^2$ and proved that it has non-zero magnetization at sufficiently low temperatures. As the model (without external field) must have zero magnetization at sufficiently high temperatures, he was the first to show that a model from statistical mechanics exhibits a phase transition. A few years later, Lars Onsager computed the critical temperature for the zero-field Ising model on the square lattice exactly and rigorously.

To simulate a configuration of the Ising model on a finite graph with given inverse temperature $\beta$, the standard approach is to use the Monte Carlo method based on the well-known algorithm by *Metropolis–Hastings*. In this rejection sampling algorithm, applied to the Ising model, one starts with an arbitrary configuration and then performs single spin updates according to the following rule: Pick a site uniformly at random and flip its spin with probability $\min\{e^{-\beta\,\Delta H}, 1\}$, where $\Delta H$ is the invoked change of the total energy. In the ferromagnetic regime without external field, flipping the spin at a chosen site might be rejected only if the majority of its neighbors agrees with the current spin as this implies $\Delta H > 0$. Evidently, a low temperature will considerably favor flips decreasing the energy over flips increasing it and therefore drive the system towards more ordered states with growing patches of aligned spins.

A different way to incorporate the microscopic evolution in a ferromagnet at a fixed temperature with help of the Ising model is the so-called *Glauber dynamics*. In this algorithm, to flip the randomly chosen spin has probability $\frac{1}{1+e^{\beta\,\Delta H}}$. In contrast to the Metropolis–Hastings algorithm, here even transitions to lower energy states might be rejected, but the tendency to order remains as updates towards lower energy have probability larger than $\frac{1}{2}$, towards higher energy less than $\frac{1}{2}$.

In a long chain of atoms, these alignments at low temperature do take place as well, but for any temperature above absolute zero, thermal fluctuations will consistently break the aligned parts of the chain and in this way prevent a global alignment of the system. This is the reason why the model on $\mathbb{Z}$ does not achieve a global magnetization even for low temperatures.

## 3.2 Sociophysics

In a colloquium in 1969, physicist Wolfgang Weidlich suggested to compare the interactions within a group of individuals holding opposing attitudes towards a given yes-no question with ferromagnetism, more precisely the dynamics of the Metropolis–Hastings algorithm applied to the Ising model. Two years later, he published this idea in the article *'The statistical description of polarization phenomena in society'* in which he elaborated how this mathematical model intended to describe and explain ferromagnetism with help of statistical mechanics can be put into a sociological context: In the sociological reinterpretation, the interaction strength of spins in the Ising model corresponds to the willingness of an individual to adopt the attitude of the majority among its neighbors and the temperature as a model parameter for the social pressure exerted on each individual (low temperature corresponding to high social pressure). An external magnetic field is understood to shape some preference of one attitude over the other, shared by all individuals.

In 1982, Galam et al. [21] used the Ising model on $K_n$, the complete graph on $n$ vertices, to describe the collective behavior in a plant where dissatisfied workers might start a strike. Using a mean field theory approach, they rediscovered the phase transition described in the foregoing section and interpreted the regime of high temperature as an individual phase (mutual influences are very limited) and low temperature as a collective one (the group behaves coherently), separated by a critical phase in which small changes in the system can lead to drastic changes in the group. In contrast to the physical application of the Ising model, where a collection of atoms is forming a regular lattice, it is reasonable to consider the underlying interaction network among workers in a small plant to be all-to-all, meaning that every worker can actually influence all his fellow workers.

Following these seminal papers, an increasing number of related models were introduced, motivated and analyzed – in the past three decades predominantly with the help of computer simulation. The principle interaction rules diverged slowly but surely from particle physics and today the area of sociophysics comprises an abundance of models for opinion dynamics in groups. For more details and further references check the comprehensive survey article *'Statistical physics of social dynamics'* by Castellano, Fortunato and Loreto [9].

## 3.3 Voter model and bounded confidence

Shortly after Weidlich's sociological reinterpretation of the Ising model, in 1973, the so-called *Voter model* was introduced by Clifford and Sudbury [11] as a model for two spatially competing species and later named for its natural interpretation in the context of opinion dynamics among voters. Its definition is very simple: Each individual holds an opinion given by a $\{-1,+1\}$-valued variable, in the standard version these are determined by i.i.d. (but not necessarily fair) coin flips to start with. At every time step, one individual is selected uniformly at random and will then adopt the opinion of another agent, picked uniformly among its neighbors. This definition of interaction apparently fails on an infinite graph but can be mimicked by introducing i.i.d. Poisson clocks at all sites governing the chronology of interactions.

On regular lattices the evolution of this model is to some extent similar to the Ising model – in one dimension, that is on $\mathbb{Z}$, it actually corresponds exactly to the limiting case of the Ising model with zero temperature. Based on well known results about random walks on grids, Clifford and Sudbury were able to conclude that on the integer lattice in dimension $d \in \{1, 2\}$ any fixed finite subset of agents will a.s. finally agree (on one of the two opinions), while this does not hold for $d \geq 3$. This behavior comes from the fact that a simple random walk on the lattice is recurrent (i.e. will a.s. return to its starting point) in dimension 1 (see exercise 18) and 2, but transient (i.e. the event that there is no return to the starting point has non-zero probability) in dimension 3 and higher.

The model introduced by Deffuant et al. [16] in 2000, features a different realistic component: When two individuals meet, they will only influence each another if their current opinion values are not too far apart from each other. More precisely, there exists a parameter $\theta \geq 0$ shaping the *tolerance* of the individuals: If the current opinion value of an agent is $\eta$, other agents holding opinions at a distance larger than $\theta$ from $\eta$ will just be ignored.

Besides the tolerance $\theta$, this model features another parameter, $\mu \in (0, \frac{1}{2}]$, that embodies the willingness of an individual to approach the opinion of the other in a compromise. Encounters always happen in pairs, so if agents $u$ and $v$ meet at time $t$, holding opinions $a, b \in \mathbb{R}$ respectively, the update rule reads as follows:

$$(\eta_t(u), \eta_t(v)) = \begin{cases} (a + \mu(b - a), b + \mu(a - b)) & \text{if } |a - b| \leq \theta, \\ (a, b) & \text{otherwise,} \end{cases}$$

where $\eta_t(u)$ denotes the opinion of agent $u$ at time $t$. The idea behind this is simple: When two individuals interact and discuss the topic in question, they will only rate the opinion encountered as worth considering if it is close enough to their own personal belief. If this is the case, however, they will have a constructive debate and their opinions will symmetrically get closer to each other – in the special case $\mu = \frac{1}{2}$, they will separate having come to a complete agreement at the average of the opinions they hold before the interaction.

In this manner, groups of compatible agents concentrate more and more around certain opinion values (their initial average) and once each such cluster of individuals is sufficiently far from neighboring ones, the final opinions are formed and all groups will from then on only become more homogeneous by internal interactions.

When Deffuant, Neau, Amblard and Weisbuch introduced this model in [16], it was considered on a finite number of agents having i.i.d. initial opinions, distributed uniformly on $[0,1]$. As social network they chose the complete graph and a finite square lattice respectively. The encounters occurred in discrete time by picking at each time step a pair of agents uniformly at random from the edge set of the underlying interaction network graph. Depending on the values of the model parameters, $\theta$ and $\mu$, in their simulation-studies they observed one of the following two long-time scenarios: Either the agents ended up in one opinion cluster (corresponding to a consensus) or split into several clusters (fragmentation or disagreement).

The first result for the Deffuant model considered on an infinite graph was published by Lanchier [40] in 2011. He studied the standard Deffuant model (i.i.d. unif($[0,1]$) initial opinions) on $\mathbb{Z}$ and was able to prove the following result using intricate geometric arguments:

**Theorem 3.9**
*Consider the Deffuant model on the graph $\mathbb{Z}$. If $\mu \in (0, \frac{1}{2}]$ is arbitrary but fixed, the initial opinions are i.i.d.* unif($[0,1]$) *and $\{\eta_t(v)\}_{v\in\mathbb{Z}}$ denotes the opinion profile at time $t$, then the following holds:*

*(i) For $\theta > \frac{1}{2}$, all neighbors are eventually compatible in the sense that for all $v \in \mathbb{Z}$:*

$$\lim_{t\to\infty} \mathbb{P}(|\eta_t(v) - \eta_t(v+1)| \leq \theta) = 1.$$

*(ii) For $\theta < \frac{1}{2}$, with probability $1$ there will be infinitely many $v \in \mathbb{Z}$ with*

$$\lim_{t\to\infty} |\eta_t(v) - \eta_t(v+1)| > \theta.$$

One thing that is quite remarkable about this phase transition in the behavior of the Deffuant model is the fact that it already occurs for the one-dimensional lattice – in marked contrast to the Ising model.

Häggström [30] used different techniques to reprove and slightly sharpen this result – showing that in the consensus regime (i), all opinions actually converge almost surely to the mean $\frac{1}{2}$ of the initial distribution. The crucial idea in his proof resides in the connection of the opinion dynamics of the Deffuant model to a non-random interaction process, which he proposed to call *Sharing a drink* (SAD):

Glasses are put, one at each vertex: the one at site $r \in V$ is full, all others are empty. As time proceeds neighbors interact and share. To be more precise, the procedure starts with the initial profile $\xi_0 = \delta_r$, i.e. $\xi_0(r) = 1$ and $\xi_0(v) = 0$ for all $v \neq r$. In each step, an edge is selected along which the two incident vertices share their water in the same way as in the Deffuant model itself (without bounded confidence restriction though): If the update is on $e = \langle u, v \rangle$ it leads to

$$\begin{aligned} \xi_{n+1}(u) &= (1-\mu)\,\xi_n(u) + \mu\,\xi_n(w), \\ \xi_{n+1}(w) &= \mu\,\xi_n(u) + (1-\mu)\,\xi_n(w), \\ \xi_{n+1}(v) &= \xi_n(v), \text{ for all } v \notin \{u, w\}. \end{aligned}$$

For arbitrary $n \in \mathbb{N}_0$, the result $\{\xi_n(v)\}_{v \in V}$ of $n$ updates involving non-empty glasses applied to $\xi_0 = \delta_r$, will be called an *SAD-profile*. Note that these profiles have only finitely many non-zero values, which are all positive and sum to 1. With $d(u, v)$ denoting the graph distance between vertices $u$ and $v$ on $G$, the following upper bound was recently extended from trees to general graphs by Huang et al. [24] using a simple but clever combinatorial argument:

**Theorem 3.10**
*Consider the SAD-process on an arbitrary graph $G = (V, E)$ started in vertex $r$, i.e. with $\xi_0(v) = \delta_r(v),\ v \in V$. Then the amount at any fixed vertex $v \in V$ is bounded from above by $\frac{1}{d(r,v)+1}$.*

The SAD-procedure is dual to the opinion formation in the sense that it keeps track of the opinion genealogy of an individual, i.e. the contributions of all initial opinions to the current composition of its opinion in the following sense:

**Lemma 3.11 (Duality)**
*Consider an initial profile $\{\eta_0(v)\}_{v \in V}$ on a finite graph $G = (V, E)$, together with a sequence $\phi$ of edges encoding the update steps, and fix a vertex $r \in V$. For $n = n(t) \in \mathbb{N}_0$ we define the SAD-process dual to the update sequence $\phi$*

*as follows: Starting with $\xi_0 = \delta_r$ the profile is updated according to but with respect to $\overleftarrow{\phi} = (e_n, \dots, e_1)$, i.e. in reversed order. Then it holds*

$$\eta_t(r) = \sum_{v \in V} \xi_t(v)\, \eta_0(v).$$

In their analyses of the Deffuant model on $\mathbb{Z}$ featuring i.i.d. unif([0, 1]) initial opinions, both Lanchier [40] and Häggström [30] singled out agents that are cast-iron centrists. These agents start with an opinion value close to the mean $\frac{1}{2}$ and will never move far away from it (irrespectively of future interactions), due to the fact that the influences they can possibly be exposed to are – loosely speaking – either close to the mean as well or marginal. The opinion $\eta_t(v)$, of an agent $v \in \mathbb{Z}$ at a later time $t > 0$, is a convex combination of all initial opinions and the maximally possible contributions on $\mathbb{Z}$ decay inversely proportional to the graph distance. Hence, the initial opinion profile $\{\eta_0(v)\}_{v\in\mathbb{Z}}$ can be such that agent $v$ sits well-shielded in a large section of individuals equipped with initial opinions close to $\frac{1}{2}$ and all individuals holding more extreme opinions are too far away to have a significant influence on $v$.

With this idea in mind (leaving aside the fact that the bounded confidence restriction might actually eliminate possible influences), obvious candidates for vertices of this kind are what Häggström [30] calls two-sidedly $\epsilon$-flat vertices and Lanchier [40] denotes by the random set

$$\Omega_0 = \Big\{ v \in \mathbb{Z};\ \tfrac{1}{2} - \epsilon < \tfrac{1}{n+1} \sum_{u=v}^{v+n} \eta_0(u),\ \tfrac{1}{n+1} \sum_{u=v-n}^{v} \eta_0(u) < \tfrac{1}{2} + \epsilon,\ \forall\, n \geq 0 \Big\}.$$

If the initial opinions are i.i.d. unif([0, 1]), it can be verified that the set $\Omega_0$ is almost surely non-empty (in fact of infinite cardinality) for all $\epsilon > 0$ (see Prop. 1.1 in [40] or Lemma 4.3 in [30]) and that the opinion at two-sidedly $\epsilon$-flat vertices will be confined to the interval $[\frac{1}{2} - 6\epsilon, \frac{1}{2} + 6\epsilon]$ for all times (see Lemma 6.3 in [30]). This consideration, however, is adjusted to the geometry of the underlying network $\mathbb{Z}$ and does not answer the question whether on more general graphs as well (e.g. higher-dimensional grids), we can find vertices whose opinions are constrained to stay close to the mean by the initial profile already.

In the standard Deffuant model, the existence of agents that will hold an opinion close to the mean $\frac{1}{2}$, no matter how the random interactions take place, force a supercritical behavior of the system (for $\theta$ sufficiently large) as they will always be at speaking terms with the whole range of opinions $[0,1]$ then.

This idea could in fact be employed to generalize the result for the Deffuant model on $\mathbb{Z}$ to initial opinion configurations other than i.i.d. unif$([0,1])$ (see [31] and [34] or seminar talk next week).

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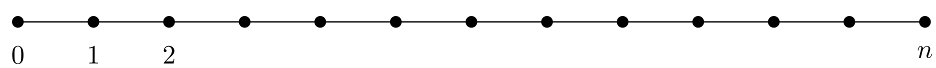
**Exercise 15**
How does the Gibbs measure on a finite graph $G=(V,E)$ look like for $\beta \to 0$ and for $\beta \to \infty$ respectively?

**Exercise 16**
Consider the path $G$ on $n+1$ vertices:




To understand what the Gibbs measure on a path looks like, we can recognize it as a simple Markov chain $(X_i)_{0 \le i \le n}$ with initial value $X_0 \sim \text{unif}\{-1,1\}$ and transition matrix

$$P = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}, \quad \text{for some } p \in [\tfrac{1}{2}, 1].$$

To this end, choose $p = \frac{1}{1+\mathrm{e}^{-2\beta}}$ and verify that $\nu_G^\beta$ then corresponds to the joint distribution of $(X_0, \ldots, X_n)$, with $Z_G^\beta = \frac{2}{p^n}$.

**Exercise 17**
Show (inductively on $|E_\eta| = n$) that the probability of an edge configuration in stage (b) to survive the conditioning in stage (c) is $\frac{2^{k(\eta)}}{2^{|V|}}$.

Also check that a spin configuration in (a) survives step (c) with probability $(1-p)^{\#\{\text{disaligned neighbors}\}}$.

**Exercise 18 (Voter model)**
Try to understand why the voter model on $\mathbb{Z}$ translates to independent annihilating random walk on the edges (the probability of a particle starting at a given edge at time 0 being $2p(1-p)$, where $p$ is the probability of a node to

be of type 1 at the start). Can you argue why locally the configuration becomes homogeneous (all nodes agree) as $t \to \infty$, however, the global density of type 1 nodes stays at $p$?

**Exercise 19**
Prove the duality lemma (3.11) rigorously.
*Hint:* Show its statement inductively on the number $n$ of moves.

# 4 Maker-Breaker games

Maker-Breaker games are a class of combinatorial games in which two players, called *Maker* and *Breaker*, compete by selecting elements from a finite or infinite structure, with opposing objectives. These games are well-studied in the context of graph theory, where Maker seeks to build a particular substructure while Breaker tries to prevent its formation. In the two most common variants of the game, either nodes or edges are played, a distinction which (depending on the objective) can become irrelevant in case the underlying graph is e.g. a tree.

Historically seen, a game of this kind was reportedly first formulated and formalized by C.E. Shannon at mid-20th century and later coined as “Shannon switching game” [41]. Day and Falgas-Ravry ([15]) extended this two-player combinatorial game to infinite graphs, in particular to the grid $\mathbb{Z}^d$ and asked natural connectivity questions such as: Will Breaker succeed in isolating the origin in a finite component or can Maker prevent that? In recent years, the study of Maker-Breaker games has been extended to boards given by random infinite structures as well.

To begin with, let us lay out the basic rules of the *Maker-Breaker* game, taking place on a graph $G = (V, E)$ – in some cases rooted, i.e. with one node marked as the origin $\mathbf{0}$. All edges in $E$ are available for play until either fixated by Maker or removed by Breaker. The two opposing players in turns are allowed to pick a fixed number of the remaining edges in order to either fixate (Maker) or remove them (Breaker). In the *connectivity* variant of the game, the objective for Breaker is to isolate the origin in a finite component, while Maker tries to prevent that. We will consider simple infinite graphs (i.e. no loops or multiple edges, $|V| = \infty$) with finite degrees and consider the game won for Breaker once the origin is contained in a finite component separated from the rest of the graph by removed edges. Maker wins if Breaker doesn't (which having finite degrees, by a simple compactness argument, translates to the task of fixating an infinite path starting at the origin). We assume both Maker and Breaker to play optimally in the sense that they try to maximize their chance of winning in every move. Note that the game is deterministic on a given graph $G$ and only becomes a game of chance if played on a random graph or by integrating randomness into the players' strategies. Further, the probability of Maker winning the game is monotonous in $G$ in the sense that adding nodes and/or edges to the graph can not decrease it. Note that this probability also might heavily depend on who starts the game.

As a first example, consider the two-dimensional grid $\mathbb{Z}^2$ (see Figure 5 for an illustration).

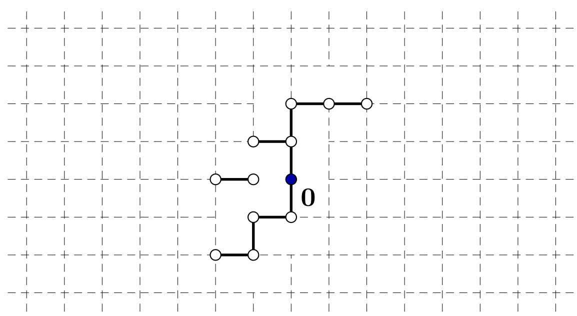



Figure 5: Illustrating example of a game position on $\mathbb{Z}^2$ after 10 moves each.

It takes little ingenuity to come up with a winning strategy for Maker in the $(1,1)$-game, i.e. when each player gets to choose one edge at a time: For every node $u = (x, y) \in \mathbb{Z}^2$, let us pair the two edges connecting $u$ to the right and upwards, i.e. to the nodes $(x+1, y)$ and $(x, y+1)$ respectively. Note that these pairs are disjoint for different $u$. Maker's strategy is then as follows: If Breaker removes a paired edge, Maker fixates the other in the subsequent turn. In this way, in fact all vertices (in particular the origin) stay connected to each other. For obvious reasons, this kind of strategy is commonly referred to as *pairing strategy*.

In order to make it more interesting, one can consider the more general $(m, b)$-variant of the game (Breaker removes $b$ edges, then Maker fixates $m$ edges in turns). Obviously, a winning strategy for Maker will persist when increasing $m$ (as it does for Breaker when $b$ is raised). However, already the outcome of the $(m, b)$-game on $\mathbb{Z}^2$ for $b = 2$ is far from trivial to determine. Note that e.g. the $(1, 1)$- and the $(2, 2)$-variant of the game are not equivalent in any sense.

## 4.1 Variants of the game

Since its introduction, an abundance of variants, differing either in the specific rules of the game, the players' objectives or the underlying graphs, have been looked at and analyzed. One of the first results [41] deals with the objective for Maker (called "short" there) to connect two given nodes on a general graph, while Breaker (called "cut") tries to prevent that. Other common objectives coined variants such as the "connectivity game", "perfect matching game", "Hamiltonian game" or "clique game", in which Maker tries to claim a spanning tree, a perfect matching, a Hamiltonian cycle or a clique of a given size respectively. Besides the rule that edges are picked one by one alternatingly, the more general $(m, b)$-rule (commonly called *biased* if $m \neq b$) has been considered and analyzed in many different contexts. Obviously, also on a random board the probability of winning is monotone in the bias for each player. Similar to a phase transition in $p$ marking the emergence of a certain structure (e.g. spanning tree, perfect matching, Hamiltonian cycle) w.h.p. in the Erdős-Rényi random graph model $G_{n,p}$ as $n$ grows large, one can investigate the corresponding threshold for the bias to allow Maker fixating such a structure (which has been done in many different settings, see below).

Chvátal and Erdős [10] considered in their seminal paper among others the biased $(1, b)$ connectivity game on the complete graph $K_n$ and found that the threshold for the bias is around $b = \frac{n}{\log n}$ as $n$ grows large (in the sense that if each round first Breaker picks $b$ edges, then Maker picks one edge, it is a Maker's win for $b$ smaller and a Breaker's win for $b$ larger than this threshold). More than 20 years later, Bednarska and Łuczak [4] analyzed the size of the largest component Maker is able to build in this context.

In the context of random boards, the Erdős-Rényi graph $G_{n,p}$ was unsurprisingly one of the first targets addressed. Stojaković and Szabó [46] established for different objectives (connectivity, perfect matching, Hamiltonian and clique game) the threshold for the edge probability $p$, at which Maker wins the unbiased $(1, 1)$-game with high probability as $n \to \infty$. In addition to that, they investigate the critical bias $b$ (asymptotically and depending on $p$) at which Maker wins the $(1, b)$-game.

In a first publication Day and Falgas-Ravry [14] considered the task of crossing a finite rectangular grid from left to right, a special case of Shannon's original game but extended to the general (potentially biased) $(m, b)$-rule.

## 4.2 On $\mathbb{Z}^d$ and its infinite percolation cluster

When it comes to the connectivity game on the $d$-dimensional grid, Day and Falgas-Ravry collected a handful interesting results in a second publication [15]. To begin with, they showed that Maker has a winning strategy for the $(m, m)$-game on $\mathbb{Z}^d$, as long as $m \leq d - 1$, adapting the pairing strategy we described above for the $(1, 1)$-game on $\mathbb{Z}^2$ (see exercise 20). In two dimensions, and with Maker starting the game, they were able to establish that the player who is allowed to pick double as many edges as their opponent has a winning strategy:

**Theorem 4.1**
*For the Maker-Breaker connectivity game on $\mathbb{Z}^2$ in which Maker starts, it holds:*

*(i) Maker wins the $(2b, b)$-game for any $b \in \mathbb{N}$ and*

*(ii) Breaker wins the $(m, 2m)$-game for any $m \in \mathbb{N}$.*

Their line of argument uses the self-duality of $\mathbb{Z}^2$, as well as a clever comparison to so-called $q$-double-response games, for which there were useful results readily adaptable.

Also in this article, among other things, they asked who wins the simplest versions not covered by their results for $\mathbb{Z}^2$, such as $(2, 2)$, $(2, 3)$ and $(3, 2)$, in general for better bounds on the critical bias in Thm. 4.1, and the question whether or not Maker can win the $(1, 1)$-game, if it was played not on the full square lattice, but on the bond percolation cluster instead (aiming for a phase transition in $p$: For $p \leq p_c(2) = \frac{1}{2}$, Breaker wins a.s. as there are no infinite components, for $p = 1$, Maker has a winning strategy. Also the board, hence Maker's chance to win, is monotone in $p$).

Inspired by these results and open questions, rather recently, Dvořák et al. [19] analyzed the connectivity game on the infinite random board given by i.i.d. bond percolation on $\mathbb{Z}^2$: Once each edge has been removed independently with a given probability $1 - p$, Maker is allowed to choose the vertex $v_0$ and tries to prevent Breaker from isolating it in a finite component. The advantage of letting Maker choose the "origin" $v_0$ is two-fold: On the one hand, this makes the game more interesting, as any given vertex can already be isolated or a pretty hopeless task for Maker to begin with, and on the other hand, it makes the game configuration translation invariant again (it would not be if the origin was fixed). This is crucial to make "Maker has a winning strategy" a translation invariant/tail event (hence having either probability 0 or 1). They showed that Breaker a.s. has a winning strategy for every $p < 1$, so that this phase transition is in fact a trivial one. Their proof readily generalizes to the $(1, d-1)$-game on bond percolation in $d$ dimensions, i.e. on $\mathbb{Z}^d$.

For the biased $(2, 1)$-game on $\mathbb{Z}^2$, they were able to prove that this phase transition happens at a non-trivial probability $p$ of an edge to be kept:

**Theorem 4.2**
*For the $(2, 1)$-Maker-Breaker connectivity game on i.i.d. bond percolation with parameter $p$ on $\mathbb{Z}^2$, it holds:*

*(i) Breaker wins a.s. if $p < 0.52784$ and*

*(ii) Maker wins a.s. if $p > 0.94013$.*

The key ingredient in their proof strategy is a potential-based approach, in which Breaker in each round evaluates the current position of the game in terms of how close Maker is to escape to infinity and plays the "most urgent" edges to keep this danger function as low as possible.

As the edges removed in the bond percolation before the game, in principle are given to Breaker as a start bonus, they introduced a natural, more symmetric generalization, which they propose to call *boosted game*: Before the start of the game, each edge is independently given to Maker with probability $\alpha$, to Breaker with probability $\beta$ and available for play with probability $1-(\alpha+\beta)$, resulting in a subgraph $(\mathbb{Z}^2)_{\alpha,\beta}$ on which Maker and Breaker can claim the unclaimed edges as before. Thm. 4.2 then refers to the $(2, 1)$-game on $(\mathbb{Z}^2)_{0,p}$.

**Definition 5**
Given an infinite vertex-transitive graph $G$, let $\mathcal{P}_n$ be the set of self-avoiding walks of length $n$ starting from a fixed vertex. Since $|\mathcal{P}_{m+n}| \le |\mathcal{P}_m|\,|\mathcal{P}_n|$, we know that $|\mathcal{P}_n|^{\frac{1}{n}}$ converges to a constant $\kappa$ (by Fekete's Lemma applied to $\log|\mathcal{P}_n|$). This constant $\kappa(G)$ is called the *connective constant* of graph $G$.

While the connective constant of $\mathbb{Z}^2$ is not known exactly, the upper bound $\kappa := \kappa(\mathbb{Z}^2) \le 2.6792$ determined by Pönitz and Tittmann in 2000 can be used to translate the following two results into regions of the parameter space (which is the triangle $\{(\alpha,\beta) \in [0,1]^2 : \ \alpha + \beta \le 1\}$ here), where either Maker or Breaker has a winning strategy.

**Theorem 4.3**
*Breaker a.s. has a winning strategy for the boosted $(m,b)$-game on $(\mathbb{Z}^2)_{\alpha,\beta}$ if*

$$1 - \alpha - \beta < (b+1)^{\frac{1}{m}}\,(\tfrac{1}{\kappa} - \alpha).$$

**Theorem 4.4**
*Maker a.s. has a winning strategy for the boosted $(m,1)$-game on $(\mathbb{Z}^2)_{\alpha,\beta}$ if $m \ge 2$ and*

$$1 - \alpha - \beta < (m+1)\,(\tfrac{1}{\kappa} - \beta).$$

These theorems (1.7 and 1.8 in [19]) are non-trivial improvements over earlier results about the Maker-Breaker game on the infinite percolation cluster on $\mathbb{Z}^2$ (as they extend into the interior of the parameter space). For the boosted $(1,1)$-game on $\mathbb{Z}^2$, only Thm. 4.3 applies (as $m = 1$), the condition there reads $\beta > 1 - \frac{2}{\kappa} + \alpha$ and together with the bound on $\kappa$ gives the following picture for the known regions in the phase diagram:

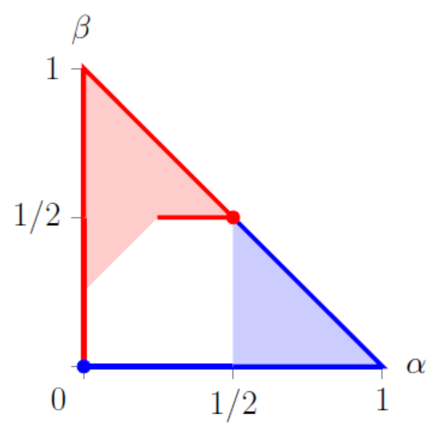



Figure 6: Known regions of the phase diagram for the $(1,1)$-game on $(\mathbb{Z}^2)_{\alpha,\beta}$:
Almost surely, Breaker wins for points coloured red and Maker wins for points coloured blue (from [19]).

For the simplest biased boosted games, $(2, 1)$ and $(1, 2)$ in two dimensions, the corresponding regions in the phase diagrams derived from Theorems 4.3 and 4.4 look like this:

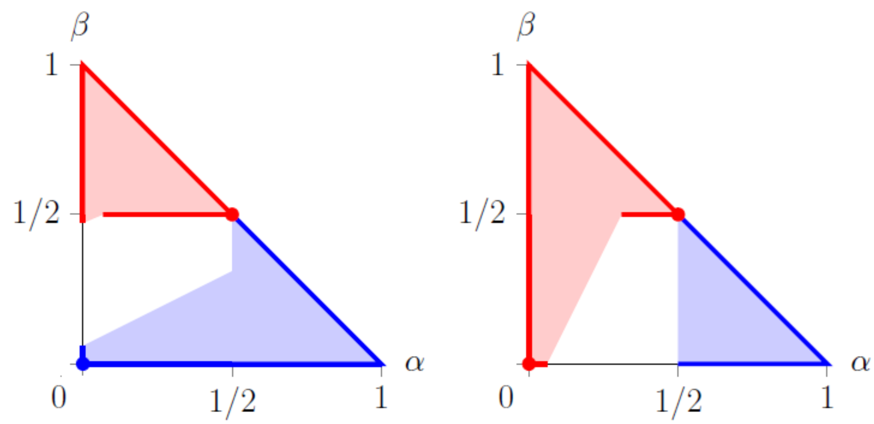



Figure 7: Known regions of the phase diagram for the $(2, 1)$-game on the left, and for the $(1, 2)$-game on the right, both played on $(\mathbb{Z}^2)_{\alpha,\beta}$ (from [19]).

For fixed $m, b \in \mathbb{N}$, by monotonicity in both directions (horizontal and vertical), there is a contour $\varphi : [0, 1] \to [0, 1]$ defined as

$$\varphi(\alpha) = \sup\{\beta \in [0, 1] : \text{ Maker a.s. wins the } (m, b)\text{-game on } (\mathbb{Z}^2)_{\alpha,\beta}\}.$$

Since $\varphi$ is non-decreasing, it can only have (upward) jump discontinuities. To determine in which cases $\varphi$ is continuous, let alone its exakt form, is still a wide open question even for the simplest choices of $m, b$.

Both articles, the one by Day and Falgas-Ravry [15] and the one by Dvořák et al. [19], contain interesting results for the infinite $d$-regular tree $\mathbb{T}_d$. In [48] the original $(1, 1)$-game (without boost) was analyzed on the random family tree of a Galton-Watson-Bienaymé branching process, focussing on different information regimes, i.e. where the board is not revealed to the players at the start necessarily, but explored incrementally as the game proceeds.

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**Exercise 20**
Devise a pairing strategy that guarantees a win for Maker in the $(m, m)$-game on the full grid $\mathbb{Z}^d$, as long as $m \leq d - 1$. Does it matter which player starts the game? Why does this strategy break down when $m \geq d$?

# References


[1] Alves, O., Machado, F. and Popov, S., Phase transition for the frog model, *Electronic Journal of Probabillity*, Vol. 7, pp. 1-21, 2002.

[2] Alves, O., Machado, F. and Popov, S., The shape theorem for the frog model, *Annals of Applied Probability*, Vol. 12, pp. 533-546, 2002.

[3] Alves, O., Machado, F., Popov, S., and Ravishankar, K., The shape theorem for the frog model with random initial configuration, *Markov Processes and Related Fields*, Vol. 7, pp. 525-539, 2001.

[4] Bednarska, M. and Łuczak, T., Biased positional games and the phase transition, *Random Structures & Algorithms*, Vol. 18, pp. 141-152, 2001.

[5] van den Berg, J. and Keane, M., A counterexample to a conjecture of J.M. Hammersley and D.J.A. Welsh concerning first-passage percolation, *Advances in Applied Probability*, Vol. 15, pp. 465-467, 1983.

[6] van den Berg, J. and Keane, M., On the continuity of the percolation probability function, *Contemporary Mathematics*, Vol. 26, pp. 61-65, 1984.

[7] Biggins, J.D. The asymptotic shape of the branching random walk, *Advances in Applied Probability* Vol. 10, pp. 62-84, 1978.

[8] Burton, R. M. and Keane, M., Density and uniqueness in percolation, *Communications in Mathematical Physics*, Vol. 121 (3), pp. 501-505, 1989.

[9] Castellano, C., Fortunato, S. and Loreto, V., Statistical physics of social dynamics, *Reviews of Modern Physics*, Vol. 81, pp. 591-646, 2009.

[10] Chvátal, V. and Erdős, P., Biased positional games, *Annals of Discrete Mathematics*, Vol. 2, pp. 221-229, 1978.

[11] Clifford, P. and Sudbury, A., A model for spatial conflict, *Biometrika,* Vol. 60 (3), pp. 581-588, 1973.

[12] Comets, F. and Popov, S., On multidimensional branching random walks in random environment, *Annals of Probability* Vol. 35, pp. 68-114, 2007.

[13] Cox, J.T., and Durrett, R., Some limit theorems for percolation processes with necessary and sufficient conditions, *Annals of Probability*, Vol. 9, pp. 583-603, 1981.

[14] Day, A.N. and Falgas-Ravry, V., Maker-Breaker percolation games I: Crossing grids, *Combinatorics, Probability and Computing*, Vol. 30, pp. 200-227, 2021.

[15] Day, A.N. and Falgas-Ravry, V., Maker-Breaker percolation games II: Escaping to infinity, *Journal of Combinatorial Theory (Series B)*, Vol. 151, pp. 482-508, 2021.

[16] Deffuant, G., Neau, D., Amblard, F. and Weisbuch, G., Mixing beliefs among interacting agents, *Advances in Complex Systems*, Vol. 3, pp. 87-98, 2000.

[17] Deijfen, M., Hirscher, T. and Lopes, F., *Electronic Journal of Probability* Vol. 24, pp. 1-17, 2019.

[18] Deijfen, M. and Vilkas, T., Competition on $\mathbb{Z}^d$ driven by branching random walk, *Electronic Communications in Probability* Vol. 28, pp. 1-11, 2023.

[19] Dvořák, V., Mond, A. and Souza, V., The Maker-Breaker percolation game on a random board, to appear in *Annals of Probability*. *arXiv:* 2402.17547

[20] Etheridge, A., Survival and extinction in a locally regulated population, *Annals of Applied Probability* Vol. 14, pp. 188-214, 2004.

[21] Galam, S., Gefen, Y. and Shapir, Y., Sociophysics: A new approach of sociological collective behaviour. I. Mean-behaviour description of a strike, *Journal of Mathematical Sociology,* Vol. 9, pp. 1-13, 1982.

[22] Garet, O. and Marchand, R., Coexistence in two-type first-passage percolation models, *Advances in Applied Probability*, Vol. 15, pp. 298-330, 2005.

[23] Garet, O. and Marchand, R., First-passage competition with different speeds: positive density for both species is impossible, *Electronic Journal of Probability*, Vol. 13, pp. 2118-2159, 2008

[24] Gollin, J.P., Hendrey, K., Huang, H., Huynh, T., Mohar, B., Oum, S., Yang, N., Yu, W. and Zhu, X., Sharing tea on a graph. *arXiv:* 2405.15353

[25] Grimmett, G., *"Percolation (2nd edition)"*, Springer, 1999.

[26] Hammersley, M. and Welsh, D.J.A., First-passage percolation, subadditive processes, stochastic networks and generalized renewal theory., *Bernoulli, Bayes, Laplace Anniversary Volume (eds. J. Neyman and L. Le Cam)*, pp. 61-110. Springer, 1965.

[27] Hara, T. and Slade, G., Mean field behavior and the lace expansion, *Probability Theory and Phase Transitions (ed. G. Grimmett)*, Proceedings of the NATO ASI meeting in Cambridge 1993, pp. 87-122, 1994.

[28] Harris, T.E., A lower bound on the critical probability in a certain percolation process, *Mathematical Proceedings of the Cambridge Philosophical Society*, Vol. 56, pp. 13-20, 1960.

[29] Häggström, O., Random-cluster representations in the study of phase transitions, *Markov Processes and Related Fields*, Vol. 4, pp. 275-321, 1998.

[30] Häggström, O., A pairwise averaging procedure with application to consensus formation in the Deffuant model, *Acta Applicandae Mathematicae*, Vol. 119 (1), pp. 185-201, 2012.

[31] Häggström, O. and Hirscher, T., Further results on consensus formation in the Deffuant model, *Electronic Journal of Probability* Vol. 19, pp. 1-26, 2014.

[32] Häggström, O.,and Pemantle, R., First passage percolation and a model for competing spatial growth, *Journal of Applied Probability*, Vol. 35, pp. 683-692, 1998.

[33] Häggström, O.,and Pemantle, R., Absence of mutual unbounded growth for almost all parameter values in the two-type Richardson model, *Stochastic Processes and Applications*, Vol. 90, pp. 207-222, 2000.

[34] Hirscher, T. The Deffuant model on $\mathbb{Z}$ with higher-dimensional opinion spaces. *Latin American Journal of Probability and Mathematical Statistics* Vol. 11, pp. 409-444, 2014. .

[35] Howard, C.D. Models of first-passage percolation, *Probability on discrete structures (ed. H. Kesten)*, Encyclopaedia of Mathematical Sciences, Vol. 110, pp. 125–173. Springer, 2004.

[36] Janson, S., An Upper Bound for the Velocity of First-Passage Percolation, *Journal of Applied Probability*, Vol. 18, pp. 256-262, 1981.

[37] Kallenberg, O., *"Foundations of Modern Probability (2nd edition)"*, Springer, 2002.

[38] Kesten, H., The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$, *Communications in Mathematical Physics*, Vol. 74, pp. 41-59, 1980.

[39] Kesten, H., Aspects of first-passage percolation, *in École d'Été de Probabilités de Saint Flour XIV,* Vol. 1180 of Lecture Notes in Mathematics, pp. 125-264, Springer, 1986.

[40] Lanchier, N., The critical value of the Deffuant model equals one half, *Latin American Journal of Probability and Mathematical Statistics*, Vol. 9 (2), pp. 383-402, 2012.

[41] Lehman, A., A solution of the Shannon switching game, *Journal of the Society for Industrial and Applied Mathematics*, Vol. 12, pp. 687-725, 1964.

[42] Liggett, T.M., An improved subadditive ergodic theorem, *Annals of Probability*, Vol. 13, pp. 1279-1285, 1985.

[43] Liggett, T.M., *"Interacting Particle Systems"*, Springer, 1985.

[44] Lyons, R. and Peres, Y., *"Probability on Trees and Networks"*, Cambridge University Press, 2017.

[45] Shi, Z., Branching Random Walks, *in École d'Été de Probabilités de Saint Flour XLII,* Vol. 2151 of Lecture Notes in Mathematics, Springer, 2015.

[46] Stojaković, M. and Szabó, T., (2005): Positional games on random graphs, *Random Structures & Algorithms*, Vol. 26, pp. 204-223, 2005.

[47] Telcs, A. and Wormald, N., Branching and tree indexed random walks on fractals, *Journal of Applied Probability*, Vol. 36, pp. 999-1011, 1999.

[48] Vilkas, T., Maker-Breaker games on Galton-Watson trees. *arXiv:* 2412.08334